$A \not\subseteq B \Rightarrow \exists x \in A$ s.t. $x \notin B$. $\therefore \forall a \in A \quad a < x$.

$x \notin B \Rightarrow \exists q \in \downarrow x$ s.t. $q \notin B$. $\therefore \forall a \in A \quad a \in \downarrow x \subset B$.

$\Rightarrow A \subseteq B$.

$\boxed{\text{Why is } \mathbb{Q}^+ \text{ \& } \mathbb{Q} \text{ countable?}}$

1

- Let $FS\mathbb{Z} = \{0\} \cup \{$ finite sequences of integers which do not end in $0\}$.

<u>define</u> $Comp: FS\mathbb{Z} \to \mathbb{Q}^+$ $\quad Comp(0) = 1$, $Comp(z_1, \dots, z_k)$

$= 2^{z_1} 3^{z_2} 5^{z_3} \cdots p_k^{z_k}$, where p_k is the k^{th} prime in $\mathbb{N}$.

The fundamental thm of arith. says Comp is 1-1 & onto:

$$q \in \mathbb{Q}^+ \Rightarrow q = \frac{n}{m}. \quad n = p_1^{i_1} \cdots p_{k_n}^{i_k}, \; m = p_1'^{j_1} \cdots p_\ell'^{j_\ell}$$

$$\therefore q = p_1'^{-j_1} \cdots p_\ell'^{-j_\ell} p_1^{i_1} \cdots p_k^{i_k}.$$ (after appropriate reordering of p''s & p's.)

The inverse of Comp is $Decomp: \mathbb{Q}^+ \to FS\mathbb{Z}$

$$Decomp(1) = 0, \quad Decomp(n/m) = (-j_1, \dots, -j_\ell, i_1, \dots, i_k)$$

$\therefore |FS\mathbb{Z}| = |\mathbb{Q}^+|$.

- Similarly $\widetilde{Comp}: FS\mathbb{N}_0 \to \mathbb{N}$, $\quad \widetilde{Comp}(0) = 1$,

$$\widetilde{Comp}((n_1, n_2, \dots, n_k)) = 2^{n_1} \cdot 3^{n_2} \cdot \dots \cdot p_k^{n_k}$$

is a <u>bijection</u> with $\widetilde{Decomp}(1) = 0$, $\widetilde{Decomp}(m) = (n_1, n_2, \dots, n_k)$

if $m = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$.

②

- Let $\xi : \mathbb{N}_0 \to \mathbb{Z}$ be the function

$$\xi(0)=0, \quad \xi(1)=1, \quad \xi(2)=-1, \; \xi(3)=2, \; \xi(4)=-2, \ldots$$

i.e. $\xi(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ even} \\ \frac{n+1}{2} & \text{if } n \text{ odd} \end{cases}$

Then ξ is a bijection with inverse: $\xi^{-1}(z) = \begin{cases} -2z & \text{if } z<0 \\ 2z-1 & \text{if } z>0 \\ 0 & \text{if } z=0 \end{cases}$

$\therefore \; |\mathbb{N}_0| = |\mathbb{Z}|.$

- Let $X\xi : FS\,\mathbb{N}_0 \to FS\,\mathbb{Z}$ be the map:

$$X\xi(\emptyset) = 0, \quad X\xi((n_1, \ldots, n_k)) = (\xi(n_1), \ldots, \xi(n_k)).$$

Then $X\xi$ has an inverse:

$$X\xi^{-1} : FS\,\mathbb{Z} \to FS\,\mathbb{N}_0 :$$

$$X\xi^{-1}(0)=0, \quad X\xi^{-1}(z_1, \ldots, z_k) = (\xi^{-1}(z_1), \ldots, \xi^{-1}(z_k)).$$

$\therefore$ $X\xi$ is a bijection ; $|FS\,\mathbb{N}| = |FS\,\mathbb{Z}|.$

$\therefore$ we have

$$\mathbb{N} \xrightarrow{\widetilde{Decomp}} FS\,\mathbb{N}_0 \xrightarrow{X\xi} FS\,\mathbb{Z} \xrightarrow{Comp} \mathbb{Q}^+.$$

The composition is a bijection.

$$\therefore \; |\mathbb{N}| = |\mathbb{Q}^+|$$

$\mathbb{Q}^+$ is countable. Let $\phi : \mathbb{N} \xrightarrow{\sim} \mathbb{Q}^+$ be the composition $\phi = Comp \circ X\xi \circ Decomp.$

- Let $g : \mathbb{N}_0 \to \mathbb{N}$ be the map $g(m) = m+1$.

  Then g is a bijection since it has inverse $g^{-1}(m) = m-1$.

  $\therefore |\mathbb{N}_0| = |\mathbb{N}|$.

- We have bijections : $\mathbb{N} \xrightarrow{g^{-1}} \mathbb{N}_0 \xrightarrow{g} \mathbb{Z}$.

  $\therefore |\mathbb{N}| = |\mathbb{Z}| \Rightarrow \mathbb{Z}$ is countable.

- note: $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$; $\mathbb{Q}^- = \{-q \mid q \in \mathbb{Q}^+\}$

  ; note: $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \mathbb{N}^-$ where $\mathbb{N}^- = \{-n \mid n \in \mathbb{N}\}$

  define: $\psi : \mathbb{Z} \to \mathbb{Q}$ to be

$$\psi(0) = 0, \quad \psi(z) = \begin{cases} \phi(z) & ; z \in \mathbb{N} \\ -\phi(z) & ; z \in \mathbb{N}^- \end{cases}$$

  Since ϕ is a bijection, ψ is a bijection.

  $\therefore |\mathbb{Z}| = |\mathbb{Q}|$; $\mathbb{Q}$ is countable since $\mathbb{Z}$ is countable.

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or: ...