Lecture 11.10.09

- If X is an infinite set, then $\exists$ 1-1 map from X to X which is not onto.

PF First: we build a sequence $\{a_i\}_{i=1}^{\infty}$ of distinct elts of X

(We do this by induction.

① $X \neq \phi$ since X is infinite. So $\exists\, x \in X$.)

let $a_1 = x$. This defines a_1, & $\forall i < 1$ $a_i \neq a_1$.

② Let $n \in \mathbb{N}$ & assume $a_1, a_2, \ldots, a_{n-1}$ have been defined & $\forall i \leq n-1$ $a_i \neq a_j$ $\forall j < i$. (this means all the a_i are distinct)

Let $A_{n-1} = \{a_1, \ldots, a_{n-1}\}$. This set is finite $\therefore$ $X \setminus A_{n-1} \neq \phi$ since X is infinite. $\therefore \exists x \in X \setminus A_{n-1}$. Let $a_n = x$. This defines a_n & $\forall i < n$ $a_n \neq a_i$.

Hence $\forall n \in \mathbb{N}$ the n^{th} term a_n is defined & the terms $\{a_1, \ldots, a_n\}$ are distinct.

Second: construct a function $g: X \to X$ that is 1-1 but not onto.

For $x \in X$, define $g(x) = \begin{cases} x & \text{if } \forall n\ x \neq a_n \\ a_{n+1} & \text{if } x = a_n. \end{cases}$ I claim g is 1-1:

Suppose $x, y \in X$ & $g(x) = g(y)$. Case 1: $x \neq a_n$ $\forall n \in \mathbb{N}$.

Then $x = g(x) = g(y)$. If $y = a_n$ then $g(y) = a_{n+1} = x \Rightarrow$ contradiction

$\therefore y \neq a_n$ $\forall n \in \mathbb{N} \Rightarrow g(y) = y$ $\therefore y = x$. Case 2: $x = a_n$ for some $n \in \mathbb{N}$. Then $a_{n+1} = g(a_n) = g(y)$. Clearly $\exists i$ s.t. $y = a_i$ (otherwise we get contradiction.) $\therefore g(y) = g(a_i) = a_{i+1} = a_{n+1}$. →

② But the a_i's are distinct by construction!
$\therefore$ we have a contradiction & only case 1 holds
which implies g is injective.

Claim: g is <u>not</u> onto:

Suppose $\exists\, x \in X$ s.t. $g(x) = a_1$. x is either equal to some a_i for some $i \in \mathbb{N}$ or <u>not</u>. If $x = a_i$ then $g(a_i) = a_{i+1} = a_1$. Since a_i are distinct & i cannot be less than 1 this is a contradiction. $\therefore \forall n \in \mathbb{N}$ $x \neq a_n$.
$\therefore g(x) = x$ by definition of g which implies $x = g(x) = a_1$
$\therefore x = a_1 \Rightarrow$ contradiction! Hence no such x exists
$\therefore$ g not onto. $\square$

• <u>Claim</u>: If X is a set, there is <u>no</u> <u>onto</u> <u>map</u> from X to $P(X)$.

<u>Pf</u> (by contradiction) Assume $\exists\, f: X \to P(X)$ s.t. f is onto.
By def. of powerset f assigns an <u>element</u> $x \in X$ to a <u>subset</u> $f(x) \subseteq X$.

$$\text{Let } S = \{x \in X \mid x \notin f(x)\}.$$

remember $f(x)$ is a <u>subset of</u> <u>X</u>

S is clearly a subset of X. f is onto. $\therefore \exists\, y \in X$ s.t. $f(y) = S$. Now $y \in S$ <u>or</u> $y \notin S$. If $y \in S$, then $y \notin f(y)$ by def. of S. But $f(y) = S \Rightarrow$ Contradiction. If $y \notin S$ then $y \notin f(y)$ $\therefore$ by def of S $y \in S = f(y) \Rightarrow$ Contradiction.

$\therefore f$ cannot be onto $\square$