Lecture 10/26/09

- **def:** (successor) Let A be a set. The successor of A is the set $A^+ = A \cup \{A\}$.

- **def:** (Inductive set) A set ω is a inductive (or successor) set iff $\phi \in \omega$ & if $n \in \omega$ then $n^+ \in \omega$.

- **Axiom** (in Zermelo-Fraenkel Set theory): There is an inductive set.

Problem #2: Let A be an inductive set.

(i) claim: if B is an inductive set then $B \cap A \neq \phi$.

PF] A & B are inductive sets $\therefore$ $\phi \in A$ & $\phi \in B$ (by def.)

$\therefore$ $\phi \in A \cap B$ by def. of intersection.

$\therefore$ $A \cap B$ contains at least 1 element

$\therefore$ $A \cap B$ is not empty $\Rightarrow$ $\phi \neq A \cap B$ $\square$

(ii) claim If $\mathcal{I}$ is a family of inductive sets then $\bigcap \mathcal{I}$ is an inductive set.

PF] By definition $\bigcap \mathcal{I} = \{x \mid x \in A \ \forall A \in \mathcal{I}\}$.

Let $A \in \mathcal{I}$. A inductive $\therefore$ $\phi \in A$

$\therefore$ $\forall A \in \mathcal{I}$, $\phi \in A$.

$\therefore$ by definition $\phi \in \bigcap \mathcal{I}$.

Let $n \in \bigcap \mathcal{I}$. By definition $\forall A \in \mathcal{I}$ $n \in A$.

Let $A \in \mathcal{I}$. Then $n \in A$. A inductive.

$\therefore$ $n^+ \in A$. $\therefore$ $\forall A \in \mathcal{I}$ $n^+ \in A$

$\therefore$ by def. $n^+ \in \bigcap \mathcal{I}$ $\therefore$ $\bigcap \mathcal{I}$ is an inductive set $\square$

②

(iii) Define $\tilde{\omega} = \bigcap \{B \subseteq A : B \text{ is inductive}\}$.
$= \{x \mid x \in B \ \forall B \subseteq A \ \& \ B \text{ inductive}\}$.

Claim: Let C be an inductive set. Then $\tilde{\omega} \subseteq C$.

Pf By (ii) $\tilde{\omega}$ is inductive. By (ii) $A \cap C$ is inductive. (take your family $\mathcal{I}$ to be $\{A, C\}$, Then $\bigcap \mathcal{I} = A \cap C$.)

Let $x \in A \cap C$. Then $x \in A$ & $x \in C$ by def of $\cap$.
$\therefore A \cap C \subseteq A$ & $A \cap C \subseteq C$ by def of $\subseteq$.

Let $x \in \tilde{\omega}$. Then $x \in B \ \forall B \subseteq A$ & B inductive.
$A \cap C \subseteq A$ & $A \cap C$ inductive $\therefore x \in A \cap C$.
Since $A \cap C \subseteq C$, $x \in C$. $\therefore \forall x \in \tilde{\omega}, x \in C$.
$\therefore \tilde{\omega} \subseteq C$. $\square$

Problem #3:

Claim: If P is linearly ordered & $f: P \to P$ is strictly increasing (i.e. $p < q \Rightarrow f(p) < f(q)$) then f is 1-1.

pf let p & $q \in P$. Assume $f(p) = f(q)$.
Since P is linearly ordered, $p < q$ or $q < p$ or $p = q$.

case 1: Assume $p < q$. Then $f(p) < f(q)$
Since $f(p) = f(q)$, $f(p) < f(p)$
Contradiction.

Case 2: Assume $q < p$. Then $f(q) < f(p)$

Since $f(q) = f(p)$, $f(q) < f(q)$

Contradiction.

$\therefore p = q$. $\therefore f$ 1-1 $\square$.

Problem #4: Let $\{P_i\}$ be the set of all Ponds that have ever existed.

Let $P_i(t)$ be the set of ducks on pond P_i at time t. Let

$\therefore R = \{(x,y) \in D \mid \exists i \text{ \& } t \text{ s.t. } x \in P_i(t) \text{ \& } y \in P_i(t)\}$.

(i) reflexive: WTS $(x,x) \in R$ $\forall x \in D$.

$\therefore$ WTS $\exists i$ & t s.t. $x \in P_i(t)$. A priori no such i & t may exist. (if they did then R is reflexive)

Not provable.

(ii) symmetric: Assume $(x,y) \in R$. $\therefore \exists i$ & t s.t. $x \in P_i(t)$ &

$y \in P_i(t) \Rightarrow \exists i$ & t s.t. $y \in P_i(t)$ & $x \in P_i(t)$

$\therefore (y,x) \in R$.

By definition R antisymmetric $\Leftrightarrow$ (if $(x,y) \in R$ &

$(y,x) \in R$ then $x = y$). Since R is symmetric,

$(x,y) \Leftrightarrow (y,x)$. $\therefore$ $\forall i$ $\forall t$ if x & $y \in P_i(t)$.

then $x = y$. Who knows? Not provable.

(iii) transitive: Assume (x,y) & $(y,z) \in R$.

then $\exists i$ & t & i' & t' s.t. x & $y \in P_i(t)$

& $y, z \in P_{i'}(t')$. But doesn't imply

$\exists i''$ & t'' s.t. x & $z \in P_{i''}(t'')$.

④

Problem #5: Let $A \subseteq \mathbb{Q}$ be an infinite set. Claim:

$\exists$ bijection $f: \mathbb{Q} \to A$

Pf $\mathbb{Q}$ countable $\therefore$ $\exists$ bijection $g: \mathbb{Q} \xrightarrow{\sim} \mathbb{N}$.

Let $g(A) = \{ g(a) \mid a \in A \}$.

Let $g|_A : A \to g(A)$ be the function $g|_A(a) = g(a)$.

Then $g|_A$ is a bijection (g is 1-1 & $g|_A$ is clearly onto.)

$\therefore$ $g(A)$ is an infinite, hence unbounded subset of $\mathbb{N}$ $\therefore$ $\exists$ a bijection $h: \mathbb{N} \to g(A)$.

Since $g|_A$ is a bijection its inverse $g|_A^{-1}: g(A) \to A$ is a bijection.

$\therefore$ we have the composition of bijections:

$$\mathbb{Q} \xrightarrow{g} \mathbb{N} \xrightarrow{h} g(A) \xrightarrow{g|_A^{-1}} A$$

Let $f = g^{-1}|_A \circ h \circ g : \mathbb{Q} \to A$.

The composition of bijections is a bijection.

$\square$