def: (successor) Let $A$ be a set. The successor of $A$ is the set $A^+ = A \cup \{A\}$.

def: (inductive set) A set $\omega$ is an inductive (or successor) set iff $\emptyset \in \omega$ \& if $n \in \omega$ then $n^+ \in \omega$.

Axiom (in Zermelo-Fraenkel set theory): There is an inductive set.

Problem #2: Let $A$ be an inductive set.

(i) Claim: If $B$ is an inductive set then $B \cap A \neq \emptyset$.

Proof: $A, B$ are inductive sets. $\emptyset \in A, \emptyset \in B$ (by def.)

$\therefore \emptyset \in A \cap B$ by def. of intersection.

$\therefore A \cap B$ contains at least 1 element

$\therefore A \cap B$ is not empty $\Rightarrow \emptyset \neq A \cap B \Box$

(ii) Claim: If $I$ is a family of inductive sets then $\cap I$ is an inductive set.

Proof: By definition $\cap I = \{x \mid \forall A \in I \ x \in A\} \cap \cap I$.

Let $A \in I$. $A$ inductive $\therefore \emptyset \in A$

$\therefore \forall A \in I, \emptyset \in A$

$\therefore$ by definition $\emptyset \in \cap I$.

Let $n \in \cap I$, By definition $\forall A \in I \ n \in A$

Let $A \in I$. Then $n \in A$. $A$ inductive

$\therefore n^+ \in A$ $\therefore \forall A \in I \ n^+ \in A$

$\therefore$ by definition $n^+ \in \cap I$ $\therefore \cap I$ is an inductive set $\Box$
(iii) Define \( \bar{\omega} = \bigcap \{B \leq A : B \text{ is inductive}\} \)
\[= \{x \mid x \in B \land B \leq A : B \text{ inductive}\}.\]

Claim: Let \( C \) be an inductive set. Then \( \bar{\omega} \subseteq C \).

Proof: By (ii) \( \bar{\omega} \) is inductive. By (ii) \( A \cap C \)

is inductive. (take your family \( \mathcal{A} \) to be \( \{A \cap C, \) Then \( \bigcap \mathcal{A} = A \cap C \).

Let \( x \in A \cap C \). Then \( x \in A \cap x \in C \) by def of \( \bigcap \).

\[A \cap C \subseteq A \cap (A \cap C) \text{ by def of } \subseteq.
\]

Let \( x \in \bar{\omega} \). Then \( x \in B \land B \leq A \) \& \( B \text{ inductive} \).

\[A \cap C \subseteq A \cap (A \cap C) \text{ inductive} \implies x \in A \cap C.
\]

Since \( A \cap C \subseteq C \), \( x \in C \). \( \forall x \in \bar{\omega}, x \in C \).

\[\therefore \bar{\omega} \subseteq C. \qed\]

Problem 3:

Claim: If \( P \) is linearly ordered \& \( f : P \rightarrow P \)

is strictly increasing (i.e. \( p < q \Rightarrow f(p) < f(q) \))

then \( f \) is 1-1.

Proof: Let \( p \neq q \in P \). Assume \( f(p) = f(q) \).

Since \( P \) is linearly ordered, \( p \leq q \) or \( q \leq p \) or \( p = q \).

Case 1: Assume \( p < q \). Then \( f(p) < f(q) \).

Since \( f(p) = f(q) \), \( f(p) < f(p) \).

Contradiction.
**Case 2:** Assume \( q < p \). Then \( f(q) < f(p) \).

Since \( f(q) = f(p) \), \( f(q) < f(q) \)

contradicting.

\[ \therefore p = q. \quad \therefore f \text{ is 1-1.} \square \]

**Problem 4:** Let \( \{ P_i \} \) be the set of all Ponds that have ever existed.

Let \( P_i(t) \) be the set of ducks on pond \( P_i \) at time \( t \). Let

\[ R = \{(x,y) \in D \mid \exists i \in t \text{ s.t. } x \in P_i(t) \land y \in P_i(t) \} \]

(i) **Reflexive:** \( \forall x \in D \), \( (x,x) \in R \).

(ii) \( \exists i \in t \) s.t. \( x \in P_i(t) \). A priori no such \( i \) may exist. (If they did then \( R \) is reflexive.)

(iii) **Symmetric:** Assume \( (x,y) \in R \). \( \exists i \in t \) s.t. \( x \in P_i(t) \)

\[ y \in P_i(t) \Rightarrow \exists i \in t \text{ s.t. } y \in P_i(t) \land x \in P_i(t) \]

\[ \therefore (y,x) \in R. \]

By definition \( R \) antisymmetric \( \Rightarrow \) (if \( (x,y) \in R \)

\( (y,x) \in R \) then \( x = y \)). Since \( R \) is symmetric,

\[ (x,y) \Leftrightarrow (y,x). \]

\( \forall i \in t \) if \( x \neq y \in P_i(t) \), then \( x = y \). Who knows? Not possible.

(iv) **Transitive:** Assume \( (x,y) \in R \) \( \land (y,z) \in R \).

Then \( \exists i \in t \) \( \land i' \in t' \) s.t. \( x \in P_i(t) \land y \in P_i(t) \land y \in P_i(t') \).

But doesn't imply

\( \exists i'' \in t'' \) s.t. \( x \in P_i(t'') \land z \in P_i(t''). \)
Problem 45: Let \( A \subseteq \mathbb{Q} \) be an infinite set. Claim:

\[ \exists \text{ bijection } f : \mathbb{Q} \to A \]

**Proof:**

\( \& \) Countable: \( \exists \) bijection \( g : \mathbb{Q} \to \mathbb{N} \).

Let \( g(A) = \{ g(a) \mid a \in A \} \).

Let \( g|_A : A \to g(A) \) be the function \( g|_A(a) = g(a) \).

Then \( g|_A \) is a bijection \( (g|_A \text{ is } 1-1 \& \text{ onto}) \).

\( \therefore g(A) \) is an infinite, hence unbounded subset of \( \mathbb{N} \), \( \exists \) a bijection \( h : \mathbb{N} \to g(A) \).

Since \( g|_A \) is a bijection, its inverse \( g|_A^{-1} : g(A) \to A \) is a bijection.

\( \therefore \) We have the composition of bijections:

\[ \mathbb{Q} \xrightarrow{g} \mathbb{N} \xrightarrow{h} g(A) \xrightarrow{g|_A^{-1}} A \]

Let \( f = g|_A^{-1} \circ h \circ g : \mathbb{Q} \to A \).

The composition of bijections is a bijection. \( \square \)