Lecture 3:

#1] Suppose $f: W \to X$ & $g: X \to Y$

s.t. $g \circ f: W \to Y$ is a bijection

<u>claim</u>: f is 1-1 & g is onto

PF] (1) f is 1-1: WTS $\forall\, w_1, w_2 \in W$ if $f(w_1) = f(w_2)$

then $w_1 = w_2$:

let $w_1, w_2 \in W$.

Assume $f(w_1) = f(w_2)$.

Then $g(f(w_1)) = g(f(w_2))$

$\Rightarrow (g \circ f)(w_1) = (g \circ f)(w_2)$.

$g \circ f$ is a bijection.

$\therefore g \circ f$ is 1-1

$\therefore w_1 = w_2$

$\therefore \forall w_1, w_2$ if $f(w_1) = f(w_2)$ then $w_1 = w_2$

$\therefore f$ 1-1. $\square$

(2) g is onto: WTS $\forall y \in Y\ \exists\, x \in X$ s.t. $y = g(x)$.

Let $y \in Y$.

$g \circ f: W \to Y$ is bijection

$\therefore g \circ f$ is onto

$\therefore \exists\, w \in W$ s.t. $(g \circ f)(w) = y$.

$f: W \to X$ $\therefore$ let $x = f(w)$.

Then $g(f(w)) = g \circ f(w) = y$.

$\therefore \forall y \in Y\ \exists x$ s.t. $y = g(x)$

$\therefore g$ onto $\square$.

(2)

Suppose $f: X \to Y$ & $g: Y \to X$ s.t.
$g \circ f: X \to X$ is the identity map on X & $f \circ g: Y \to Y$ is the identity map on Y:

Claim: f is a bijection.

Pf WTS f is 1-1 & onto.

(1) (1-1): $g \circ f: X \to X$ is equal to the identity map $id_X: X \to X$.

id_X is a bijection.

$\therefore$ $g \circ f$ is a bijection.

$\therefore$ By previous claim, f is 1-1.

(2) (onto): $f \circ g: Y \to Y$ is equal to the identity map $id_Y: Y \to Y$.

id_Y is a bijection

$\therefore$ $f \circ g$ is a bijection

$\therefore$ by previous claim f is onto.

$\therefore$ f is 1-1 & onto $\therefore$ f is bijection. □

Claim: g is a bijection

Pf exercise. Follow the same steps as above ~~above~~ (with a slight twist)

#2] Define a bijection from $8^{\mathbb{N}}$ onto $2^{\mathbb{N}}$; where $8 = \{0, 1, \ldots, 7\}$
$2 = \{0, 1\}$

Step 1: Note that any number from 0 to 7 has a binary representation; If $n \in 8$ then n can be uniquely written as:

$$n = \sum_{i=0}^{2} a_i 2^i = a_0 2^0 + a_1 2^1 + a_2 2^2,$$

where each a_i is either 0 or 1. The triple (a_0, a_1, a_2) is the binary representation of n.

So:

$0 = (0,0,0)$	$4 = (0,0,1)$
$1 = (1,0,0)$	$5 = (1,0,1)$
$2 = (0,1,0)$	$6 = (0,1,1)$
$3 = (1,1,0)$	$7 = (1,1,1)$

Step 2: Define the function $\phi : 2 \times 2 \times 2 \longrightarrow 8$
where $\phi(a_0, a_1, a_2) = a_0 2^0 + a_1 2^1 + a_2 2^2$, i.e. (a_0, a_1, a_2) is the binary representation of n. From the table above ϕ is well-defined, 1-1 & onto.

Step 3: Let $f \in 2^{\mathbb{N}}$. Then $f : \mathbb{N} \to 2$. From f we can define a sequence in 8 by:

$$X_1 = f(1) 2^0 + f(2) 2^1 + f(3) 2^2$$
$$X_2 = f(4) 2^0 + f(5) 2^1 + f(6) 2^2$$
$$X_3 = f(7) 2^0 + f(8) 2^1 + f(9) 2^2$$
$$\vdots$$
$$X_n = f(3n-2) 2^0 + f(3n-1) 2^1 + f(3n) 2^2$$

$\therefore$ let $g : \mathbb{N} \to 8$ be the sequence:

$$g(n) = \phi\big(f(3n-2), f(3n-1), f(3n)\big).$$

Step 4: The map $\psi : 2^{\mathbb{N}} \to 8^{\mathbb{N}}$ defined by $\psi(f) = g$ where $g(n) = \phi\big(f(3n-2), f(3n-1), f(3n)\big)$

(4) is a bijection since ϕ is a bijection & 2 sequences f_1 & f_2 are equal if and only if $f_1(n) = f_2(n)$.

Since ψ is a bijection, $\psi^{-1}: 8^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}$ is a bijection.

Let $\tilde{\phi} = \phi^{-1}: 8 \to \mathbb{Z}$ Let $\tilde{\psi} = \psi^{-1}: 8^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}$.

$n \mapsto (a_0, a_1, a_2)$

If $f \in 8^{\mathbb{N}}$, let $x_i = f(i)$. Let $\tilde{\phi}_0(n) = a_0$, $\tilde{\phi}_1(n) = a_1$, $\tilde{\phi}_2(n) = a_2$.

Let $y_1 = \tilde{\phi}_0(f(1))$, $y_2 = \tilde{\phi}_1(f(1))$, $y_3 = \tilde{\phi}_2(f(1))$

$y_4 = \tilde{\phi}_0(f(2))$, $y_5 = \tilde{\phi}_1(f(2))$, $y_6 = \tilde{\phi}_2(f(2))$

$\vdots$

$y_n = \tilde{\phi}_i(f($

$3 \cdot 1 = \frac{3\cdot 1}{3} + 0$

$\frac{3\cdot 1}{3} + 1$

The sketch: (For exam, etc.)

(1) Let $\phi: \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to 8$ be the function: $\phi(a_0, a_1, a_2) = n$ where (a_0, a_1, a_2) is the binary representation of n. Every $n \in 8$ has a unique binary representation $\therefore$ ϕ is a bijection.

(2) Let $f \in \mathbb{Z}^{\mathbb{N}}$ be a sequence. Let $g \in 8^{\mathbb{N}}$ be the sequence

$g(1) = f(1)2^0 + f(2)2^1 + f(3)2^2$

$g(2) = f(4)2^0 + f(5)2^1 + f(6)2^2$

$\vdots$

$g(n) = f(3n-2)2^0 + f(3n-1)2^1 + f(3n)2^2 = \phi(f(3n-2), f(3n-1), f(3n))$

(3) The function $\psi: \mathbb{Z}^{\mathbb{N}} \to 8^{\mathbb{N}}$, $\psi(f) = g$ with $g(n) = \phi(f(3n-2), f(3n-1), f(3n))$ is a bijection since ϕ is a bijection & 2 sequences f_1 & f_2 are equal iff $f_1(n) = f_2(n)$.

#3] Claim: $2^{\mathbb{N}}$ is not countable:

PF] (Cantor's diagonal argument & proof by contradiction).

Assume $2^{\mathbb{N}}$ is countable.

$\therefore 2^{\mathbb{N}} = \{f_1, f_2, f_3, \ldots\}$ where $f_i : \mathbb{N} \to 2$ is a sequence.

Let $g : \mathbb{N} \to 2$ be the function: $g(n) = \begin{cases} 0 & ; \text{ if } f_n(n) = 1 \\ 1 & ; \text{ if } f_n(n) = 0. \end{cases}$

Then $g : \mathbb{N} \to 2$ is a sequence.

$\therefore g \in 2^{\mathbb{N}}$. $\therefore \exists K \in \mathbb{N}$ s.t. $g = f_K$.

$\therefore g(K) = f_K(K)$.

But if $f_K(K) = 0$, $g(K) = 1$; if $f_K(K) = 1$, then $g(K) = 0$

Hence contradiction.

$\therefore 2^{\mathbb{N}}$ is not countable □

#4] Claim: $\mathbb{N}_0^{\mathbb{N}}$ is not countable.

PF] Assume $\mathbb{N}_0^{\mathbb{N}}$ countable.

$\therefore \mathbb{N}_0^{\mathbb{N}} = \{f_1, f_2, \ldots\}$.

Let $g : \mathbb{N} \to \mathbb{N}_0$. be the sequence $g(n) = f_n(n) + 1$.

The rest is an exercise ... □
(same as #3)

#5] Claim: $FS\mathbb{N}_0 \approx FS\mathbb{Z}$.

Pf] Define the function $f : \mathbb{N}_0 \to \mathbb{Z}$ by $f(n) = \begin{cases} -\left(\frac{n+1}{2}\right) & \text{if } n \text{ odd} \\ \frac{n}{2} & \text{if } n \text{ even.} \end{cases}$

Then $f(0) = 0$, $f(1) = -1$, $f(2) = 1$

$f(3) = -2$, $f(4) = 2$, ...

Let $g : \mathbb{Z} \to \mathbb{N}_0$, be the function $g(n) = \begin{cases} 0, & n = 0 \\ -2n-1, & n < 0 \\ 2n, & n > 0 \end{cases}$,

Then $f \circ g = id_{\mathbb{Z}}$ & $g \circ f = id_{\mathbb{N}_0}$

$\therefore$ f is a bijection with inverse g (by problem #1).

$\therefore$ $\Phi : FS\mathbb{N}_0 \to FS\mathbb{Z}$

$\Phi(\{n_1, n_2, \ldots, n_k, 0 \to\}) = \{f(n_1), f(n_2), \ldots, f(n_k), 0 \to\}$

is a bijection since f is a bijection. $\therefore$ $FS\mathbb{N}_0 \approx FS\mathbb{Z}$ $\square$

#6] Claim: $\mathbb{N} \approx \mathbb{Q}^+$.

Pf] We know Decomp : $\mathbb{N} \to FS\mathbb{N}_0$ is a bijection.

& by #5 $\Phi : FS\mathbb{N}_0 \to FS\mathbb{Z}$ is a bijection.

& Comp : $FS\mathbb{Z} \to \mathbb{Q}^+$ is a bijection

$\therefore$ Comp $\circ$ Φ $\circ$ Decomp : $\mathbb{N} \to \mathbb{Q}^+$ is a bijection

(why?) $\square$

• Let A be infinite $B \subset A$ finite & $C = A \setminus B$.

then $\exists$ bijection $\phi : A \to C$:

Pf: C infinite $\therefore \exists\, \psi : \mathbb{N} \to C$ s.t. ψ is 1-1. $\therefore H = \psi(\mathbb{N})$ is a countable subset of C. We want to "squeeze" B into H. Let $B = \{b_1, \dots b_N\}$.

Let $H = \{h_1, h_2, \dots\dots\}$

$\{h_i\}$ are unique since H injective.

Define $\phi : A \to C$ by:

For $a \in$

- for $a \in C \setminus H$ $\phi(a) = a \in C \setminus H$.
- for $b_i \in B$ $\phi(b_i) = h_i \in H$.
- for $h_i \in H$ $\phi(h_i) = h_{N+i} \in H$

By construction, ϕ is 1-1. And ϕ is onto on $C \setminus H$ & ϕ maps $B \cup H$ onto H. $\therefore$ ϕ bijective.

$\therefore$ $\phi : A \to A$ is 1-1 but not onto.

• To find onto map not 1-1: Let $B \subset A$ be finite with $|B| \geq 2$. Let $x \in B$. Let $\tilde{\phi} : A \to A$ be the map: $\tilde{\phi}(a) = \phi^{-1}(a)$ if $a \in A \setminus B$
$= x$ if $a \in B$.

Since $\phi : A \to A \setminus B$ is a bijection.

$\phi^{-1} : A \setminus B \to A$ is onto

$\therefore$ $\tilde{\phi}$ is onto but not 1-1.

↑↑ IGNORE THIS ↑↑↑

$\boxed{\#0}$ Let $f: W \to X$ & $g: X \to Y$ be onto.

claim: $g \circ f: W \to Y$ is onto.

$\boxed{\text{Pf}}$ WTS $\forall\, y \in Y\ \exists\, w \in W$ s.t. $g \circ f(w) = y$.

Let $y \in Y$.

g is onto.

$\therefore \exists\, x \in X$ s.t. $g(x) = y$.

f is onto. & $x \in X$.

$\therefore \exists\, w \in W$ s.t. $f(w) = x$.

$\therefore g \circ f(w) = g(f(w)) = g(x) = y$.

$\therefore \forall y \in Y\ \exists w \in W$ s.t. $g \circ f(w) = y$

$\therefore g \circ f$ is onto. $\square$

THIS is the 1st problem we did today (problem "0")