$\mathbb{C}P^2$ is the mapping cone of the Hopf fibration

- Hopf fibration: $h: S^3 \to S^2$, $h(a,c) = (2a\bar{c}, |a|^2 - |c|^2)$

$$\begin{matrix} S^3 & \to & S^2 \\ \cap & & \cap \\ \mathbb{C}^2 & & \mathbb{C}\times\mathbb{R} \end{matrix}$$

$$= (2\operatorname{Re} a\bar{c}, 2\operatorname{Im} a\bar{c}, |a|^2 - |c|^2)$$

There exists a circle action:

$$A: S^1 \times S^3 \to S^3$$
$$(\alpha, a, c) \mapsto (\alpha a, \alpha c).$$

- Claim 1: $h: S^3 \to S^2$ is onto

Pf There are 2 circle subgroups of $\mathcal{U}(2)$:

$$\left\{ \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \middle| \, |\omega| = 1 \right\} \;\&\; \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \, \theta \in [0, 2\pi) \right\}$$

Consider:

$$h\left(\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = h(\omega\cos\theta, \sin\theta)$$

$$= (2\cos\phi\cos\theta\sin\theta, 2\sin\phi\cos\theta\sin\theta, \cos^2\theta - \sin^2\theta),$$

where $\omega = e^{i\phi}$.

Use trig identities: $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\Rightarrow h(w\cos\theta, \sin\theta) = \underbrace{\left(\sin 2\theta\cos\phi,\ \sin 2\theta\sin\phi,\ \cos 2\theta\right)}_{S^2 \text{ is spherical coords.}}$$

$\square$

- <u>Claim 2</u>: Let S^3/S^1 be the quotient space:

$$\left\{ [a,c] \;\middle|\; (a,c) \sim (a',c') \Leftrightarrow \exists\, \alpha \in S^1 \text{ s.t. } \alpha(a,c) = (a',c') \right\}.$$

Then there exists a homeomorphism $\tilde{h} : S^3/S^1 \xrightarrow{\sim} S^2$ s.t.

$$(*) \qquad \begin{tikzcd} S^3 \arrow[r, "q"] \arrow[dr, "h"'] & S^3/S^1 \arrow[d, dashed, "\tilde{h}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & S^2 \end{tikzcd} ,$$

where $q(a,c) = [a,c]$ is the canonical quotient map.

$\boxed{\text{pf}}$ The Hopf map $h : S^3 \to S^2$ respects the equivalence on S^3, i.e. $h(\alpha a, \alpha c) = h(a,c)$.

Define $\tilde{h}([a,c]) = h(a,c)$. By Claim 1, $\tilde{h}$ is onto.

Assume $h(a,c) = h(a',c')$. WTS $\exists\, \alpha \in S^1$ s.t. $\alpha a = a'$ & $\alpha c = c'$. (This will imply $\tilde{h}$ is 1-1.)

Case 1: $(a=0)$ Then by def. of h, $|c|^2 = |c'|^2 \Rightarrow$

$c = c'e^{i\theta}$, for some θ.

Case 2: $(a \neq 0)$. $\exists\, \gamma, \gamma' \in S^1$ s.t. γa & $\gamma' a'$ are real.

Since $h(a,c) = h(\gamma a, \gamma c) = h(\gamma' a', \gamma' c') = h(a', c')$,

(& S^1 is a group), we may assume WLOG that a & a'

themselves are real.

$\therefore$ By def. of h; $c = \dfrac{a'}{a} c'$, $a^2 - |c|^2 = a'^2 - |c'|^2$,

& $a^2 + |c|^2 = a'^2 + |c'|^2 = 1$.

$\therefore$ $a^2 = a'^2 \Rightarrow a = \pm a' \Rightarrow c = \pm c'$.

$\therefore$ In general, if $(\gamma a, \gamma c) = \pm(\gamma' a', \gamma' c')$, then

$(a,c) = \pm \gamma^{-1}\gamma'(a', c')$ & $\pm \gamma^{-1}\gamma' \in S^1$.

$\therefore$ $\tilde{h}$ is 1-1.

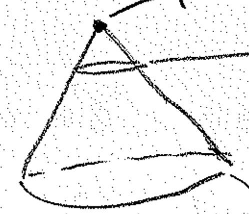
The universal property of quotient maps $\Rightarrow$ $\tilde{h}$ is continuous.

S^3 compact $\Rightarrow$ S^3/S^1 compact. S^2 Hausdorff $\Rightarrow$ $\tilde{h}$ is closed

map.

$\therefore$ $\tilde{h}$ is a homeomorphism. $\square$

• The mapping cone for $h: S^3 \to S^2$

$C_h =$ 

pt. $(t=1)$

every slice for $0 < t < 1$ is homeomorphic to $S^3 \times \{t\}$

$S^2 = h(S^3)$ ($t=0$ slice).

$\therefore$ $C_h \setminus S^2$ is homeo. to $\mathring{D}^4$ (the interior of the 4-Disk $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 \leq 1\}$)

$\therefore$ $C_h = D^4 \cup_h S^2$, where we use h to attach $\partial D^4 = S^3$ to S^2.

• <u>Claim 3</u>: $\mathbb{C}P^2 = \mathbb{C}^3 \setminus \{0\} / \sim$, where $v \sim v' \Leftrightarrow \exists z \in \mathbb{C} \setminus \{0\}$ s.t. $v = z v'$.

is homeomorphic to S^5 / S^1.

Pf. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{C}^3 \setminus \{0\} \arrow[r, "q_1"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C}P^2 \arrow[d, dashed, "\tilde{q}"] \\
S^5 \arrow[r, "q_2"] & S^5 / S^1
\end{tikzcd}$$

where q_1 & q_2 are the canon. quotient maps,

$p(z) = \dfrac{z}{\|z\|}$ & $\tilde{q}([z_1, z_2, z_3]) = [z_1, z_2, z_3]$ as a equiv. class in S^5/S^1

show $\tilde{q}$ is homeomorphism by following the proof of claim 2 $\square$

· <u>claim 4</u>: Define the map: $j : D^4 \longrightarrow S^5$,

$$j(z_1, z_2) = (z_1, z_2, (1-|z_1|^2-|z_2|^2)^{1/2}).$$

Let $\tilde{j} = q_2 \circ j : D^4 \to S^5/S^1$. Then $\tilde{j}$ is onto & $\tilde{j}|_{\mathring{D}^4}$ is 1-1.

pf (onto): Let $[z_1, z_2, z_3] \in S^5/S^1$. By choosing an appropriate representative, WMA $z_3 = x_3$ is real.

$|z_1|^2 + |z_2|^2 + x_3^2 = 1 \Rightarrow x_3 = \pm(1-|z_1|^2-|z_2|^2)^{1/2}$

& $|z_1|^2+|z_2|^2 \leq 1$. $\therefore \tilde{j}(z_1, z_2) = [z_1, z_2, z_3]$.

($\tilde{j}|_{\mathring{D}^4}$ is 1-1): Assume $\tilde{j}(z_1, z_2) = [z_1, z_2, x_3]$

$= [z_1', z_2', x_3'] = \tilde{j}(z_1', z_2')$, with $x_3 = (1-|z_1|^2-|z_2|^2)^{1/2}$,

$x_3' = (1-|z_1'|^2-|z_2'|^2)^{1/2}$. If (z_1, z_2) & $(z_1', z_2') \in \mathring{D}^4$,

then $x_3 \neq 0$ & $x_3' \neq 0$. $\exists \alpha \in S^1$ s.t. $(\alpha z_1, \alpha z_2, \alpha x_3)$

$= (z_1', z_2', x_3')$. Since $x_1, x_3 \in \mathbb{R}$ & are positive,

$\alpha = 1$. $\therefore (z_1, z_2) = (z_1', z_2')$. $\square$

• Claim 5: S^5/S^1 is homeomorphic to $D^4 \cup_h S^2$.

Pf] Let $S^3 \overset{i}{\hookrightarrow} S^5$ be the embedding $(z_1, z_2) \mapsto (z_1, z_2, 0)$.

Pass to the quotient: $S^3/S^1 \overset{\tilde{i}}{\hookrightarrow} S^5/S^1$

where $\tilde{i}([z_1, z_2]) = [z_1, z_2, 0]$.

$\therefore \tilde{j}\big|_{\partial D^4} = \{[z_1, z_2, 0] \mid |z_1|^2 + |z_2|^2 = 1\} = \tilde{i}(S^3/S^1)$.

Let $\Phi: D^4 \sqcup S^2 \to S^5/S^1$ be the map:

$$\Phi(z_1, z_2) = \begin{cases} \tilde{j}(z_1, z_2) & \text{if } (z_1, z_2) \in D^4 \\ \tilde{i} \circ \tilde{h}^{-1}(z_1, z_2) & \text{if } (z_1, z_2) \in S^2 \\ & \text{(here } z_2 \text{ is real).} \end{cases}$$

where $\tilde{h}$ is the homeomorphism (*) in Claim 2.

Then $\exists\ \tilde{\Phi}: D^4 \cup_h S^2 \to S^5/S^1$ s.t.

$$\begin{tikzcd} D^4 \sqcup S^2 \arrow[r] \arrow[dr, "\Phi"'] & D^4 \cup_h S^2 = D^4 \sqcup S^2/\sim \arrow[d, "\tilde{\Phi}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & S^5/S^1 \end{tikzcd}$$

where $(z_1, z_2) \sim (z_1', z_2')$ $\iff (z_1, z_2) \in \partial D^4$ & $(z_1', z_2') \in S^2$ & $h(z_1, z_2) = (z_1', z_2')$.

Exercise: Show $\tilde{\Phi}$ respects $\sim$ & is 1-1 & onto, so that it is a homeomorphism $\square$