Office hrs: M-W 1.30-2.30

Midterm next Friday

- Chapters 4.1-4.3, 5.1-5.2

extra office hrs Wed - 1.30-4.

Chapter 5: Eigenvalues and Eigenvectors

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slides will be on my homepage
www.math.ucr.edu/chari
available after lecture in afternoon.

A matrix – compute powers of A, A^2, A^3, ...

– this is computationally a very hard problem – substitute is to calculate the eigenvalues and eigenvectors of A.

<u>defn</u>: eigenvalue of a $n \times n$-matrix A is a number λ such that there exists vector ~~...~~ v in $\mathbb{R}^n$ such that $Av = \lambda v$

The vector v is called ~~the~~ a eigenvect of A with eigenvalue λ.

A $n \times n$-matrix v – $n \times 1$ column vector.

Av – column vector, $n \times 1$, $Av = \lambda v$

Same as saying Av and v are parallel vectors

ex: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$Av = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2v$$

- $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eig. vect. of A with eigenvalue 2.

- computing eigenvalues and eigenvectors

Compute eigenvalues of A

1. Let x be an unknown. calculate $A - xI$ I = identity $n \times n$ matrix
2. Calculate $|A - xI|$
3. Solve equation $|A - xI| = 0$ for x.

④ The values of x found in step 3 are eigenvalues of A

ex: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$: find its eigenvalues

$$A - xI = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$$

$$= \begin{bmatrix} 2-x & 0 \\ 0 & 2-x \end{bmatrix}$$

$$|A - xI| = (2-x)^2$$

$$|A - xI| = 0 \Rightarrow (2-x)^2 = 0$$

$$\Rightarrow x = 2$$

$\Rightarrow$ the eigenvalue value of A is 2 ex 23(b) T or F that

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every $n \times n$ matrix has n ~~real~~ distinct eigenvalues

- False, previous excereise is the counterexample showing false

ex 23(c) ~~every~~ There can be only one eigenvalue associated with an eigenvector of a matrix

T or F ?

True: Proof: Let v be an eigenvector of the matrix A

$\Rightarrow$ there exists λ such that

$Av = \lambda v$. — ①

Suppose v also has eigenvalue λ'

$Av = \lambda' v$ — ②

$\lambda v = \lambda' v \Rightarrow (\lambda - \lambda')v = 0$ ③

$\Rightarrow \lambda = \lambda'$

<u>ex 23 (5)</u>. λ is an eigenvalue for A then λ is an eigenvalue for $A + cI$ for all scalars c.

<u>given:</u> there exists $v \in \mathbb{R}^n$ such that $Av = \lambda v$.

<u>want:</u> is there a vector $w \in \mathbb{R}^n$ such that

$$(A + cI)w = \lambda w$$

- ~~only~~ information given. is very little - so reasonable try for w is $\underline{v}$ $(A+cI)v = \lambda v$?

$$Av + cIv = \lambda v + cv = \lambda v$$

$$\Rightarrow c = 0$$

- it is reasonable to assume statement false. look for an example which shows false.

<u>ex</u>: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

eigenvalues are 2

Let c be any non-zero scalar

eg: $c = 1$

$A + cI = A + I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

eigenvalues of $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 3-x & 0 \\ 0 & 3-x \end{bmatrix}$

$\det = (3-x)^2 = 0 \Rightarrow x = 3$

eigenvalues of $A+I$ are 3

$\Rightarrow$ 2 not an eigenvalue of $A+I$, so this is an example

Computing Eigenvectors corresponding to a given eigenvalue λ.

1. Calculate $A - \lambda I$
2. Find its null-space
3. any element of its nullspace is an eigenvector of A.

Review null-space of a matrix B.

$= \{w \in \mathbb{R}^n : Bw = 0\}$

- in other words all vectors which satisfy matrix equation

$Bw = 0$

ex: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Find the eigenvectors corresponding to the eigenvalue 2.

① $A - 2I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

② null-space of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

any element $\begin{bmatrix} x \\ y \end{bmatrix}$ are in null-space. – all elements in $\mathbb{R}^2$ are eigenvectors

T/F There can be only one eigenvector associated with a given eigenvalue – False
prev. example is a counterex

typical problem:

Find the eigenvalues and eigenvectors of a matrix A.

1. Find eigenvalues by solving equation $|A - \alpha I| = 0$
   - called the characteristic eq^n of a matrix.

2. Corresponding to each eigenvalue λ find a basis for the null-space of $(A - \lambda I)$

ex: 10

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -5 \\ 8 & 0 & 9 \end{bmatrix}$$

$$A - xI = \begin{bmatrix} 1-x & 0 & 0 \\ -8 & 4-x & -5 \\ 8 & 0 & 9-x \end{bmatrix}$$

$$|A - xI| = (1-x)\big((4-x)(9-x) - (-5)(0)\big)$$

$$= (1-x)(4-x)(9-x)$$

$$|A - xI| = 0$$

Eigenvalues:

$x = 1$, $x = 4$, $x = 9$

Next: Find eigenvectors find null-space of $A - I$, $A - 4I$ and $A - 9I$.

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ -8 & 3 & -5 \\ 8 & 0 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 8 & 0 & 8 \\ -8 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 8 & 0 & 8 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

– row echelon form.

so $(A-I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\begin{bmatrix} 8 & 0 & 8 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x + 8z = 0$$

$$3y + 3z = 0$$

$z = 1 \Rightarrow y = -1 \quad x = -1.$

Basis for null sp $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$