

Diagonals in Fell algebras, an interim report.

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Introduction.

A Fell algebra is a C^* -algebra A which satisfies Fell's condition; this is equivalent to requiring that abelian elements generate. In this case A is almost a continuous trace algebra but \widehat{A} need not be Hausdorff. Such algebras arise naturally in the study of certain dynamical systems. We prove:

- ▶ An abelian C^* -subalgebra B of a Fell algebra A is a diagonal iff it satisfies the extension property.
- ▶ Up to Rieffel-Morita equivalence (RME) each such A contains a diagonal.
- ▶ The twists arising from RME Fell algebras containing diagonals are “equivalent”.
- ▶ We define a complete invariant for Fell algebras: $\delta(A) \in \text{Br}(X)$ where A is a Fell algebra with $\widehat{A} = X$ (see [DD63]).



Fell algebras and abelian elements.

Definitions (see [Ped79]): Let A be a C^* -algebra.

- ▶ An element $a \in A_+$ is said to be *abelian* if \overline{aAa} is abelian.
- ▶ We say that A is *type I_0* if it is generated by abelian elements.

It is easy to show that a C^* -algebra A is type I_0 iff it satisfies Fell's condition: for every $x \in \widehat{A}$, there is an element $p \in A_+$ and an open neighborhood U of x such that $\pi(p)$ is a rank 1 projection for every irreducible representation π with $[\pi] \in U$. Continuous trace algebras are Fell.

Note, $C^*(\mathbb{Z} \rtimes \mathbb{Z}_2)$ is a Fell algebra with non-Hausdorff spectrum.

Suppose A is separable; A is Fell iff there are ideals J_n such that

- ▶ J_n is RME to an abelian algebra,
- ▶ the J_n generate A as an ideal.

Hence, if A is Fell, then \widehat{A} is locally Hausdorff.



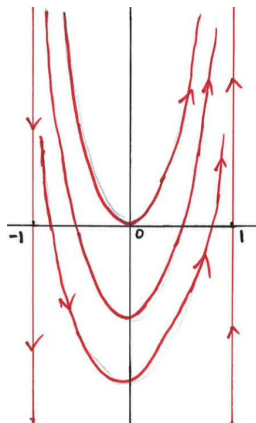
Example.

Dynamical systems which give rise to Fell algebras have been characterized in [anH01].

There is a free action of \mathbb{R} on the strip $X := [-1, 1] \times \mathbb{R}$ with non-Hausdorff quotient.

Using this characterization one sees that the crossed product $C_0(X) \rtimes \mathbb{R}$ is Fell.

Its spectrum is the quotient of two copies of $[0, 1)$ identified along $(0, 1)$.



Astrid's sketch.



Extension property.

Let B be a C^* -subalgebra of A . Following Archbold et al (see [ABG82]) we say that B has the *extension property* relative to A if

- ▶ pure states of B extends uniquely,
- ▶ B contains an approximate unit for A .

The extension property passes to quotients.

If B is an abelian C^* -subalgebra with the extension property, then

- ▶ B is a maximal abelian subalgebra of A ,
- ▶ there is a unique conditional expectation $P : A \rightarrow B$,
- ▶ for each pure state $x \in \widehat{B}$, its unique extension is given by $x \circ P$ (which is necessarily pure).



Diagonals.

Let B be an abelian C^* -subalgebra of A . An element $n \in A$ is said to be a *normalizer* of B if $nBn^*, n^*Bn \subset B$. Such an n is said to be *free* if $n^2 = 0$. The collection of free normalizers is denoted $N_f(B)$.

Definition (see [Kum86]): B is said to be a diagonal in A if

- ▶ there is a faithful conditional expectation $P : A \rightarrow B$,
- ▶ B contains an approximate unit for A ,
- ▶ $\text{span } N_f(B)$ is dense in $\ker P$.

A complete invariant for the diagonal pair (A, B) is given by an extension of groupoids with common unit space \widehat{B} ,

$$\widehat{B} \times \mathbb{T} \rightarrow \Gamma \rightarrow R,$$

sometimes referred to as the *twist invariant*.



The spectral map Ψ .

Let B be an abelian C^* -subalgebra of A which has the extension property relative to A . Given $x \in \widehat{B}$, we let $(\pi_x, \mathcal{H}_x, \xi_x)$ denote the GNS triple associated to the pure state $x \circ P$.

We define the *spectral map* $\Psi : \widehat{B} \rightarrow \widehat{A}$ by $\Psi(x) = [\pi_x]$.

Lemma: If A is liminary, Ψ is surjective.

Proof (sketch): Let π be an irreducible representation of A . Then $\pi(B)$ is a masa in $\pi(A) = \mathcal{K}(\mathcal{H})$. Hence, there exist $\{x_n\} \subset \widehat{B}$ and a family of orthogonal rank one projections $\{p_n\}$ such that

$$\pi(b) = \sum_n x_n(b)p_n \quad \text{for } b \in B.$$

Therefore, $\Psi^{-1}([\pi]) = \{x_n\}$ and Ψ is onto.



Structure Theorem.

Let B be an abelian C^* -subalgebra of A which has the extension property relative to A .

If A is Fell, then it is liminary and the previous lemma holds.

But more is true (cf. [Kum85]).

Theorem: Suppose that A is a Fell algebra. Then

- ▶ Ψ is a local homeomorphism,
- ▶ B is a diagonal in A ,
- ▶ $R = R(\Psi)$.

The theorem fails if A is only assumed to be liminary.

Note that diagonals automatically satisfy the extension property.



Using abelian elements to construct a diagonal.

Let A be a separable Fell algebra. Then A is generated by a countable family of abelian elements $\{a_n\}$. We construct a diagonal in a C^* -algebra RME to A as follows.

We let \mathcal{K} denote the compacts and let $\{e_{mn}\}$ be a family of matrix units. Set

$$a := \sum_n 2^{-n} a_n \otimes e_{nn}$$

and $C := \overline{a(A \otimes \mathcal{K})a}$. Then C is RME to A .

Moreover, the C^* -algebra D generated by elements of the form

$$a_n b a_n \otimes e_{nn} \quad \text{where } b \in A$$

is a diagonal in C .



Twist invariants of RME Fell algebras.

There were choices made in the above diagonal construction. What, if anything, can one use to derive an invariant?

Proposition: Let A_1 and A_2 be Fell algebras which are RME and let L be a linking algebra containing A_1 and A_2 . Suppose that B_i is a diagonal in A_i for $i = 1, 2$. Then $B_1 \oplus B_2$ is a diagonal in L .

Corollary: With assumptions as above, the associated twist invariants

$$\widehat{B}_i \times \mathbb{T} \rightarrow \Gamma_i \rightarrow R_i$$

are equivalent in the sense of Renault (see [Ren85], [Kum86]).

Corollary: The equivalence class of the twist obtained from the diagonal construction is independent of the choices made and only depends on the RME class of A .



An analog of the Dixmier-Douady invariant.

Let (A, B) be a diagonal pair where A is Fell algebra with $\widehat{A} = X$. We define the Brauer group $\text{Br}(X) := H^2(R(\Psi), \mathcal{S})$ where Ψ is the spectral map associated to the diagonal pair (A, B) and \mathcal{S} is the sheaf of germs of continuous \mathbb{T} -valued functions. The group $\text{Br}(X)$ does not depend on the choices made.

Let $\text{Tw}(G)$ denote the group of twists over an étale groupoid G . There is a natural map $\partial^1 : \text{Tw}(G) \rightarrow H^2(G, \mathcal{S})$.

With (A, B) as above, let Γ denote the twist invariant and define

$$\delta(A) = \partial^1([\Gamma]) \in H^2(R(\Psi), \mathcal{S}) = \text{Br}(X).$$

Two twists arising from RME equivalent diagonal pairs map to the same element of $\text{Br}(X)$.



A liminary C^* -algebra

We construct an example of a liminary algebra with an abelian subalgebra which has the extension property but is not a diagonal. Set

$$C := \{f \in C([0, 1], M_2) : f(0) \in \mathbb{C}I\},$$

and let D be the subalgebra of C consisting of functions f such that each $f(t)$ is a diagonal matrix.

Then C is liminary and D is a diagonal subalgebra which therefore satisfies the extension property with respect to C .

For $t > 0$, let

$$u_t := \begin{pmatrix} \cos(1/t) & \sin(1/t) \\ -\sin(1/t) & \cos(1/t) \end{pmatrix}.$$



Define $\alpha \in \text{Aut}(C)$ by

$$\alpha(f)(t) = \begin{cases} u_t f(t) u_t^* & \text{if } t > 0 \\ f(0) & \text{if } t = 0. \end{cases}$$








Let $A := M_2(C)$ and let B be the abelian subalgebra of matrices of the following form

$$\begin{pmatrix} d_1 & 0 \\ 0 & \alpha(d_2) \end{pmatrix}, \quad \text{where } d_1, d_2 \in D.$$

Then A is liminary and B the extension property relative to A , but B is not a diagonal subalgebra of A .



Some references.

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Thanks!
Any questions?

