Higher Gauge Theory, Division Algebras and Superstrings

John Baez and John Huerta

June 16, 2011
Quantum Theory and Gravitation

for more, see:
http://math.ucr.edu/home/baez/susy/

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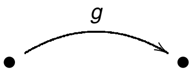

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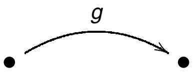

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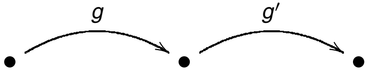

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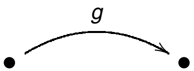

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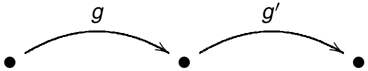
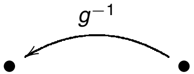
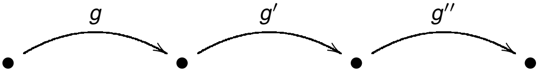


while reversing the direction corresponds to taking the inverse:



The associative law makes the holonomy along a triple composite unambiguous:



So: *the topology dictates the algebra!*

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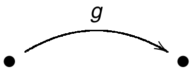

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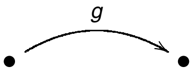
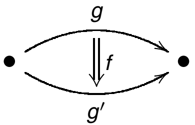


but also morphisms:



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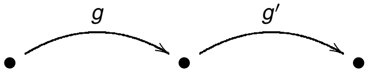

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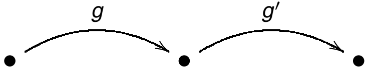

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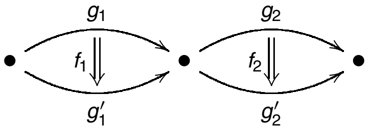

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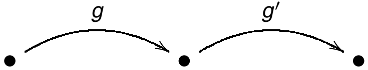

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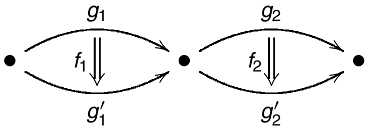
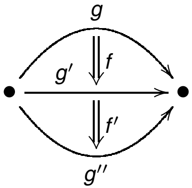


and also compose morphisms:



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Just as a group is a monoid where every element has an inverse, a **2-group** is a monoidal category where every object and every morphism has an inverse.

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A spin foam model based on this Lie 2-group may serve as a ‘quantum model of flat 4d spacetime’, much as the Ponzano–Regge model does for 3d spacetime. See:

- Aristide Baratin and Derek Wise, 2-group representations of spin foams, arXiv:0910.1542.

for the Euclidean case.

Other examples show up in string theory. In his thesis, John Huerta showed that they explain this pattern:

- The only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. They have dimensions $k = 1$, 2, 4 and 8.
- The classical superstring makes sense only in dimensions $k + 2 = 3$, 4, 6 and 10.
- The classical super-2-brane makes sense only in dimensions $k + 3 = 4$, 5, 7 and 11.

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For superstrings we need Lie 2-supergroups; for super-2-branes we need Lie **3**-supergroups.

To get our hands on Lie *n*-supergroups, it's easiest to start with 'Lie *n*-superalgebras'. Let's see what those are.

An $\boldsymbol{L_\infty}$**-algebra** is a chain complex

$$L_0 \stackrel{d}{\leftarrow} L_1 \stackrel{d}{\leftarrow} \cdots \stackrel{d}{\leftarrow} L_n \stackrel{d}{\leftarrow} \cdots$$

equipped with the structure of a Lie algebra ‘up to coherent chain homotopy’.

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Which Lie 2-superalgebras, if any, are relevant to superstrings?

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(Don't worry, soon I'll tell you what $\mathfrak{siso}(T)$ actually *is!*)

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Let's see how it works in detail.

If V is a finite-dimensional real vector space with a quadratic form Q, the **Clifford algebra** $\mathrm{Cliff}(V)$ is the real associative algebra generated by V with relations

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In the first case $S_+ \not\cong S_-$. In the second, set $S_+ = S_- = S$. In either case, let's call S_+ and S_- **left- and right-handed spinors**.

$\mathrm{Cliff}_0(V)$ acts on S_+ and S_-. But the whole Clifford algebra acts on $S_+ \oplus S_-$, with odd elements interchanging the two parts. So, we can 'multiply' a spinor by a vector and get a spinor of the other handedness:

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So, let's see when $\dim(V) = \dim(S_+) = \dim(S_-)$.

V	$S_\pm$	normed division algebra?
$\mathbb{R}^1$	$\mathbb{R}$	YES: $\mathbb{R}$
$\mathbb{R}^2$	$\mathbb{C}$	YES: $\mathbb{C}$
$\mathbb{R}^3$	$\mathbb{H}$	NO
$\mathbb{R}^4$	$\mathbb{H}$	YES: $\mathbb{H}$
$\mathbb{R}^5$	$\mathbb{H}^2$	NO
$\mathbb{R}^6$	$\mathbb{C}^2$	NO
$\mathbb{R}^7$	$\mathbb{R}^8$	NO
$\mathbb{R}^8$	$\mathbb{R}^8$	YES: $\mathbb{O}$

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$\mathbb{R}^4$	$\mathbb{H}$	YES: $\mathbb{H}$
$\mathbb{R}^5$	$\mathbb{H}^2$	NO
$\mathbb{R}^6$	$\mathbb{C}^2$	NO
$\mathbb{R}^7$	$\mathbb{R}^8$	NO
$\mathbb{R}^8$	$\mathbb{R}^8$	YES: $\mathbb{O}$

Increasing k by 8 multiplies $\dim(S_\pm)$ by 16, so these are the *only* normed division algebras!

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Now vectors V are the 2×2 Hermitian matrices with entries in $\mathbb{K}$:

$$V = \left\{ \begin{pmatrix} t+x & \overline{y} \\ y & t-x \end{pmatrix} : t, x \in \mathbb{R}, \quad y \in \mathbb{K} \right\}.$$

Now our quadratic form Q comes from the determinant:

$$\det \begin{pmatrix} t+x & \overline{y} \\ y & t-x \end{pmatrix} = t^2 - x^2 - |y|^2$$

Now right- and left-handed spinors are elements of $\mathbb{K}^2$, and 'multiplying' a vector $A \in V$ and a spinor $\psi \in S_+$ gives a spinor $A \cdot \psi = (A - \mathrm{tr}(A)1)\psi \in S_-$.

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As reps of $\mathrm{Spin}(V)$ we have

$$V^* \cong V \qquad\qquad S_+^* = S_-$$

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Concretely:

$$[\psi, \phi] = \psi\phi^\dagger + \phi\psi^\dagger$$

It's symmetric!

So, we can define the **translation Lie superalgebra**

$$T = V \oplus S_+$$

with V as its even part and S_+ as its odd part. We define the bracket to be zero except for $[-,-]\colon S_+ \otimes S_+ \to V$. The Jacobi identity holds trivially.

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The corresponding Lie supergroup acts as symmetries of 'Minkowski superspace'.

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In these dimensions the multiplication

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obey the identity

$$[\psi,\psi] \cdot \psi = 0$$

Sudbery, Chung, Manogue, Dray, Janesky, Schray, *et al* proved this with a calculation using $\mathbb{K}$-valued matrices.

We get a Poincaré Lie superalgebra whenever we have an invariant symmetric bracket that takes two spinors and gives a vector. What's so special about the dimensions 3, 4, 6 and 10?

In these dimensions the multiplication

$$\cdot\colon V \otimes S_+ \to S-$$

and bracket

$$[-,-]\colon S_+ \otimes S_+ \to V$$

obey the identity

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Sudbery, Chung, Manogue, Dray, Janesky, Schray, *et al* proved this with a calculation using $\mathbb{K}$-valued matrices.

In fact, *only* for Minkowski spacetimes of dimension 3, 4, 6, and 10 does this identity hold!

The identity $[\psi, \psi] \cdot \psi = 0$ lets us extend the Poincaré Lie superalgebra $\mathfrak{siso}(T)$ to a Lie 2-superalgebra

$$\mathfrak{siso}(T) \stackrel{d}{\leftarrow} \mathbb{R}$$

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The idea:

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- $[-, -, -]$ is zero unless two arguments are spinors and one is a vector in $\mathfrak{siso}(T)$, and

$$[\psi, \phi, v] = g([\psi, \phi], v) \in \mathbb{R}$$

where $\psi, \phi \in S_+$, $v \in V$, and $g \colon V \otimes V \to R$ is the **Minkowski metric**: the bilinear form corresponding to Q.

To get a Lie 2-superalgebra this way, the ternary bracket must obey an equation. This says that

$$[-,-,-] \colon \mathfrak{siso}(T)^{\otimes 3} \to \mathbb{R}$$

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The equation

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is this cocycle condition in disguise.

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Theorem (John Huerta)

There is a 2-group in the category of supermanifolds, $\mathrm{Superstring}(T)$*, whose Lie 2-superalgebra is* $\mathfrak{superstring}(T)$*.*

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This theorem takes real work to prove. Not every Lie 2-superalgebra has a corresponding 'Lie 2-supergroup' in such a simple-minded sense! There are important finite-dimensional Lie 2-algebras that don't come from 2-groups in the category of manifolds—instead, they come from 'stacky' Lie 2-groups.

He also went further:

Theorem (Huerta)

In Minkowski spacetimes of dimensions 4, 5, 7 and 11, we can use division algebras to construct a 4-cocycle on the Poincaré Lie superalgebra.

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In Minkowski spacetimes of dimensions 4, 5, 7 and 11, we can use division algebras to construct a 4-cocycle on the Poincaré Lie superalgebra.

This gives a Lie 3-superalgebra

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called $\mathbf{2\text{-}\mathfrak{brane}(\mathit{T})}$.

Moreover, there is a 3-group in the category of supermanifolds, $2\text{-}\mathrm{Brane}(T)$*, whose Lie 3-superalgebra is* $2\text{-}\mathfrak{brane}(T)$.

2-$\mathrm{Brane}(T)$ is relevant to the theory of supersymmetric 2-branes in dimension 4, 5, 7 and 11. And so, the octonionic case is relevant to M-theory (whatever that is).

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Summary: For theories that include gravity and describe the parallel transport of extended objects, we want Lie n-groups extending the Lorentz or Poincaré group.

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Summary: For theories that include gravity and describe the parallel transport of extended objects, we want Lie *n*-groups extending the Lorentz or Poincaré group. Lie *n*-*super*groups extending the Poincaré *super*group only exist in special dimensions, thanks to special properties of the normed division algebras. In 10 and 11 dimensions, the octonions play a crucial role.

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For more see:

- John Huerta, *Division Algebras, Supersymmetry and Higher Gauge Theory*, at http://math.ucr.ed/home/baez/susy/