Higher Gauge Theory (II)

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joint work with:
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in honor of
Ross Street's 60th birthday

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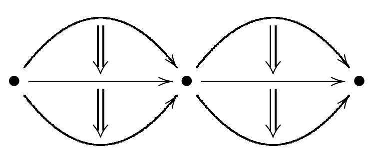


More details at:

http://math.ucr.edu/home/baez/street/

Gauge Theory

Ordinary gauge theory describes how point particles transform as they move along paths in spacetime:

In the simplest setup, a 'transformation' is an element of a smooth group G, and 'spacetime' is a smooth space M.

(We work in a convenient category of 'smooth spaces', including smooth manifolds as a full subcategory, but cartesian closed, with all limits and colimits.)

A **connection** is a $\mathfrak{g}$-valued 1-form A on M. This lets us compute a **holonomy** $\mathrm{hol}(\gamma) \in G$ for each path $\gamma \colon [0,1] \to M$, as follows. Solve this differential equation:

$$\frac{d}{dt} g(t) = A(\gamma'(t)) g(t)$$

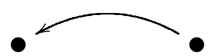
with initial value $g(0) = 1$. Then let

$$\mathrm{hol}(\gamma) = g(1).$$

Holonomy as a Functor

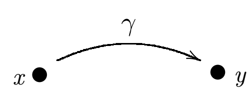
The holonomy along a path doesn't depend on its parametrization. When we compose paths, their holonomies multiply:

When we reverse a path, we get a path with the inverse holonomy:



So, let $\mathcal{P}_1(M)$ be the **path groupoid** of M:

- objects are points $x \in M$: $\bullet_x$
- morphisms are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant near $t = 0, 1$:



This is a **smooth groupoid**: it has a smooth space of objects and a smooth space of morphisms, with all groupoid operations being smooth.

Theorem. Given connection on a smooth space M, its holonomies along paths determine a smooth 'holonomy functor':

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to G.$$

Bundles

The story so far is oversimplified. It's evil to demand that holonomies *are* group elements – we should only demand that each point in M have a *neighborhood* in which holonomies *can be regarded as* group elements.

So, define a **bundle** over M to be:

- a smooth space P (the **total space**),
- a smooth space F (the **standard fiber**),
- a smooth map $p\colon P \to M$ (the **projection**),

such that for each point $x \in M$ there exists an open neighborhood U equipped with a diffeomorphism

$$f\colon p^{-1}U \to U \times F,$$

(the **local trivialization**) such that

$$\begin{array}{ccc} p^{-1}U & \xrightarrow{f} & U \times F \\ & {\scriptstyle p|_{p^{-1}U}} \searrow \quad \swarrow & \\ & U & \end{array}$$

commutes.

Principal Bundles

If F is a smooth space, $\mathrm{Aut}(F)$ is a smooth group. Given a bundle $P \to M$ with standard fiber F, the local trivializations over neighborhoods U_i covering M give:

- smooth maps (**transition functions**)

$$g_{ij} \colon U_i \cap U_j \to \mathrm{Aut}(F)$$

such that:

- $g_{ij}(x) g_{jk}(x) = g_{ik}(x)$,
- $g_{ii}(x) = 1$.

For any smooth group G, we say a bundle $P \to M$ has G as its **structure group** when the maps g_{ij} factor through an action $G \to \mathrm{Aut}(F)$.

If furthermore $F = G$ and G acts on F by left multiplication, we say P is a **principal G-bundle**.

Connections

What's a connection on a principal G-bundle $P \to M$? In each neighborhood U_i it's a $\mathfrak{g}$-valued 1-form A_i, but we demand compatibility:

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} d g_{ij}^{-1}$$

on the intersections $U_i \cap U_j$.

What is the holonomy of such a connection along a path? There is a smooth groupoid $\mathrm{Trans}(P)$, the **transport groupoid**, for which:

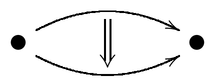
- objects are the fibers $P_x = p^{-1}(x)$ for $x \in M$, which are ***G*-torsors**: right G-spaces isomorphic to G.
- morphisms are G-torsor morphisms $f \colon P_x \to P_y$.

Theorem. Any connection on a principal G-bundle $P \to M$ gives a smooth 'holonomy functor':

$$\mathrm{hol} \colon \mathcal{P}_1(M) \to \mathrm{Trans}(P).$$

Higher Gauge Theory

Higher gauge theory should describe how strings transform as move them along surfaces in spacetime:



So, let's categorify all the above and get a theory of *2-connections on principal 2-bundles!*

The crucial trick is 'internalization'. Given a familiar gadget x and a category K, we define an 'x in K' by writing the definition of x using commutative diagrams and interpreting these in K.

We will need these examples:

- A **smooth group** is a group in [Smooth Spaces].
- A **smooth groupoid** is a groupoid in [Smooth Spaces].
- A **smooth category** is a category in [Smooth Spaces].
- A **smooth 2-group** is a 2-group in [Smooth Spaces].
- A **smooth 2-groupoid** is a 2-groupoid in [Smooth Spaces].

Here 2-groups and 2-groupoids come in two flavors: *strict* and *coherent.* In the former all laws hold as equations; in the latter, they hold up to specified isomorphisms which satisfy coherence laws of their own. For details, see my paper with Aaron Lauda and references therein.

2-Bundles

Toby Bartels has developed a theory of 2-bundles, which we roughly sketch here.

We can think of a smooth space M as a smooth category with only identity morphisms. A **2-bundle** over M consists of:

- a smooth category P (the **total space**),
- a smooth category F (the **standard fiber**),
- a smooth functor $p\colon P \to M$ (the **projection**),

such that each point $x \in M$ is equipped with an open neighborhood U and a smooth equivalence:

$$f\colon p^{-1}U \to U \times F$$

(the **local trivialization**) such that:

$$\begin{array}{ccc} p^{-1}U & \xrightarrow{f} & U \times F \\ & {}_{p|_{p^{-1}U}}\searrow \quad \swarrow & \\ & U & \end{array}$$

commutes.

Principal 2-Bundles

If F is a smooth category, $\mathcal{G} = \mathrm{Aut}(F)$ is a smooth 2-group. Given a 2-bundle $P \to M$ with standard fiber F, the local trivializations over open sets U_i covering M give:

- smooth maps

$$g_{ij} \colon U_i \cap U_j \to \mathrm{Ob}(\mathcal{G})$$

- smooth maps

$$h_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{Mor}(\mathcal{G})$$

  with

$$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \to g_{ik}(x)$$

- smooth maps

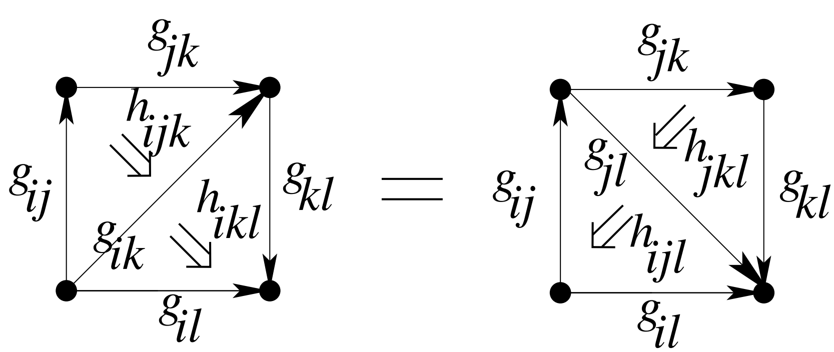
$$k_i \colon U_i \to \mathrm{Mor}(\mathcal{G})$$

  with

$$k_i(x) \colon g_{ii}(x) \to 1 \in \mathcal{G}.$$

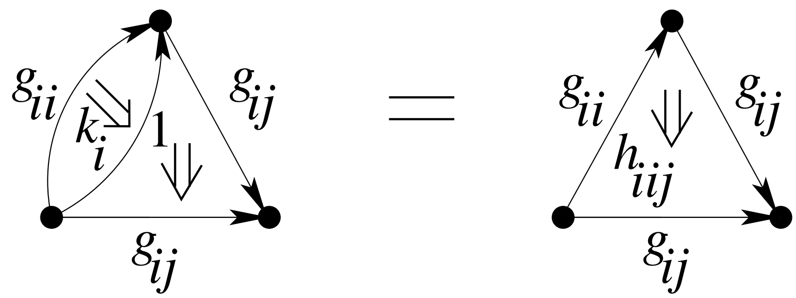
Furthermore:

- h satisfies an equation on quadruple intersections $U_i \cap U_j \cap U_k \cap U_\ell$:

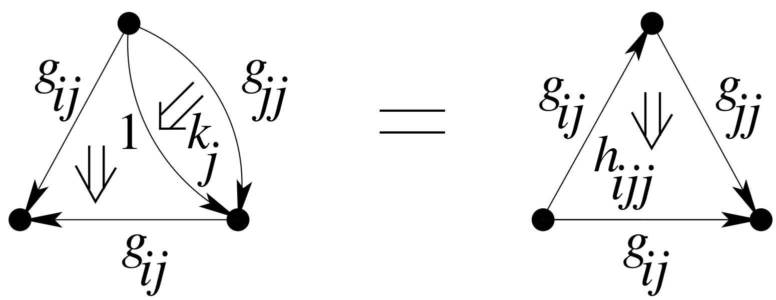


  (the **associative law**)

- k satisfies two equations on double intersections $U_i \cap U_j$:



(the **left unit law**) and



(the **right unit law**).

For any smooth 2-group $\mathcal{G}$, we say a 2-bundle $P \to M$ has $\mathcal{G}$ as its **structure 2-group** when g_{ij}, h_{ijk}, and k_i factor through an action $\mathcal{G} \to \mathrm{Aut}(F)$.

If furthermore $F = \mathcal{G}$ and $\mathcal{G}$ acts on F by left multiplication, we say P is a **principal $\mathcal{G}$-2-bundle**.

2-Connections

So far Urs Schreiber and I have only handled 2-connections on principal 2-bundles where the structure 2-group $\mathcal{G}$ is *strict*.

A smooth strict 2-group $\mathcal{G}$ is determined by:

- the smooth group G consisting of all objects of $\mathcal{G}$,
- the smooth group H consisting of all morphisms of $\mathcal{G}$ with source 1,
- the homomorphism $t\colon H \to G$ sending each morphism in H to its target,
- the action α of G on H defined using conjugation in the group $\mathrm{Mor}(\mathcal{G})$ via

$$\alpha(g)h = 1_g h 1_g{}^{-1}$$

The system (G, H, t, α) satisfies equations making it a 'crossed module'. Conversely, any crossed module of smooth groups gives a strict smooth 2-group.

Let $\mathcal{G}$ be a strict smooth 2-group.

Let (G, H, t, α) be its crossed module.

Let $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ be the corresponding 'differential crossed module' — the Lie algebra analogue of a crossed module.

If $P \to M$ is a principal 2-bundle with structure 2-group $\mathcal{G}$ and U_i is an open cover of M by neighborhoods equipped with local trivializations of P, we can describe a **2-connection** on P in terms of:

- a $\mathfrak{g}$-valued 1-form A_i on each open set U_i,
- an $\mathfrak{h}$-valued 2-form B_i on each open set U_i,

together with some extra data and equations for double and triple intersections. The details are in our paper; as we'll see, these 2-connections are closely related to Breen and Messing's *connections on nonabelian gerbes.*

If P is trivial ($P = M \times \mathcal{G}$) all this reduces to:

- a $\mathfrak{g}$-valued 1-form A on M,
- an $\mathfrak{h}$-valued 2-form B on M.

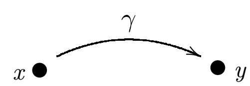
Holonomy as a 2-Functor

Let's consider a 2-connection on a trivial 2-bundle and ponder the existence of a holonomy 2-functor

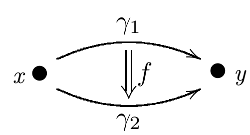
$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$$

where the **path 2-groupoid** $\mathcal{P}_2(M)$ is defined so that:

- objects are points of M: $\bullet\, x$
- morphisms are smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant in a neighborhood of $t = 0$ and $t = 1$:



- 2-morphisms are thin homotopy classes of smooth maps $f\colon [0,1]^2 \to M$ such that $f(s,t)$ is independent of s in a neighborhood of $s = 0$ and $s = 1$, and constant in a neighborhood of $t = 0$ and $t = 1$:



Recall: $\mathcal{G}$ is a strict smooth 2-group with crossed module (G, H, t, α). A 2-connection on a trivial principal $\mathcal{G}$-2-bundle over M consists of:

- a $\mathfrak{g}$-valued 1-form A on M,
- an $\mathfrak{h}$-valued 2-form B on M.

Theorem. A 2-connection on a trivial principal $\mathcal{G}$-2-bundle determines a smooth 'holonomy 2-functor':

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$$

if and only if its **fake curvature** vanishes:

$$F_A - dt(B) = 0,$$

where F_A is the usual curvature of A, namely the $\mathfrak{g}$-valued 2-form $F_A = dA + A \wedge A$.

Vanishing fake curvature guarantees that parallel transport along a surface $f\colon [0,1]^2 \to M$ is *invariant under thin homotopies* — in particular, invariant under reparametrizations of $[0,1]^2$.

All this generalizes to nontrivial principal $\mathcal{G}$-2-bundles using the **transport 2-groupoid** $\mathrm{Trans}(P)$, for which:

- objects are the fibers P_x (which are $\mathcal{G}$-2-torsors),
- morphisms are 2-torsor morphisms $f\colon P_x \to P_y$,
- 2-morphisms are 2-torsor 2-morphisms $\theta\colon f \Rightarrow g$.

Theorem. A 2-connection on a principal $\mathcal{G}$-2-bundle $P \to M$ determines a smooth 'holonomy 2-functor':

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathrm{Trans}(P)$$

if and only if the fake curvature vanishes.

2-Bundles, Stacks and Gerbes

Just as a bundle has a sheaf of sections, a 2-bundle has a 'stack of sections'. This must be defined carefully, using the local trivializations. In certain cases this stack is a gerbe!

Any smooth category X determines a smooth 2-group $\mathrm{Aut}(X)$, in which:

- objects are smooth equivalences $f\colon X \to X$.
- morphisms are smooth natural isomorphisms $\theta\colon f \Rightarrow g$.

A smooth group H is a special sort of smooth category, so it gives a smooth 2-group $\mathrm{Aut}(H)$.

Theorem. The stack of sections of a principal $\mathrm{Aut}(H)$-2-bundle $P \to M$ is a nonabelian H-gerbe. A connection on this nonabelian gerbe (in the sense of Breen and Messing) is the same as a 2-connection on P.

Toby Bartels is working on:

Conjecture. The 2-category of principal $\mathrm{Aut}(H)$-2-bundles over M is biequivalent to the 2-category of nonabelian H-gerbes over M.