Higher Gauge Theory

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joint work with:
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in honor of
Ross Street's 60th birthday

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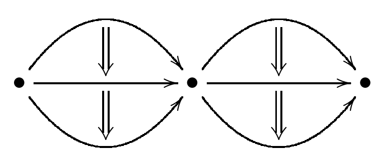
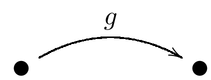


More details at:

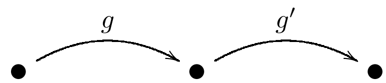
http://math.ucr.edu/home/baez/street/

Gauge Theory

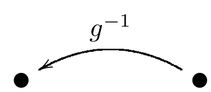
Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a Lie group element to each path:



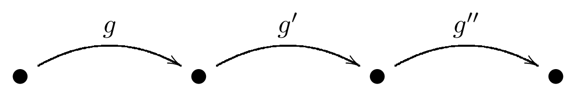
since composition of paths then corresponds to multiplication:



while reversing the direction of a path corresponds to taking inverses:



and the associative law makes the holonomy along a triple composite unambiguous:

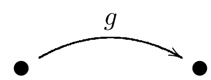


In short: *the topology dictates the algebra!*

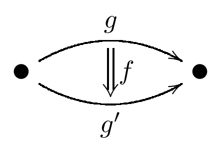
The electromagnetic field is described using the group is U(1). Other forces are described using other groups.

Higher Gauge Theory

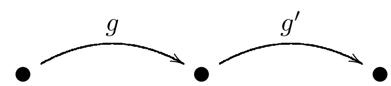
Higher gauge theory describes the parallel transport not only of point particles, but also 1-dimensional strings. For this we must categorify the notion of a group! A '2-group' has objects:



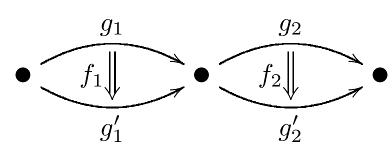
and also morphisms:



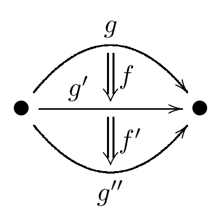
We can multiply objects:



multiply morphisms:



and also compose morphisms:



Various laws should hold....

In fact, we can make this precise and categorify the whole theory of Lie groups, Lie algebras, bundles, connections and curvature!

2-Groups

A group is a monoid where every element has an inverse. Let's categorify this!

A **2-group** is a monoidal category where every object x has a 'weak inverse':

$$x \otimes y \cong 1, \qquad y \otimes x \cong 1$$

and every morphism f has an inverse:

$$fg = 1, \qquad gf = 1.$$

A **homomorphism** between 2-groups is a monoidal functor. A **2-homomorphism** is a monoidal natural transformation. So, the 2-groups X and X' are **equivalent** if there are homomorphisms

$$f \colon \mathcal{G} \to \mathcal{G}' \qquad \bar{f} \colon \mathcal{G}' \to \mathcal{G}$$

that are inverses up to 2-isomorphism:

$$f\bar{f} \cong 1, \qquad \bar{f}f \cong 1.$$

Theorem. 2-groups are classified up to equivalence by quadruples consisting of:

- a group G,
- an abelian group H,
- an action α of G as automorphisms of H,
- an element $[a] \in H^3(G, H)$.

Lie 2-Algebras

To categorify the concept of 'Lie algebra' we must first treat the concept of 'vector space':

A **2-vector space** L is a category for which the set of objects and the set of morphisms are vector spaces, and all the category operations are linear.

We can also define **linear functors** between 2-vector spaces, and **linear natural transformations** between these, in the obvious way.

Theorem. The 2-category of 2-vector spaces, linear functors and linear natural transformations is equivalent to the 2-category of:

- 2-term chain complexes $C_1 \xrightarrow{d} C_0$,
- chain maps between these,
- chain homotopies between these.

The objects of the 2-vector space form the space C_0. The morphisms $f\colon 0 \to x$ form the space C_1, with $df = x$.

A **Lie 2-algebra** consists of:

- a 2-vector space L

equipped with:

- a functor called the **bracket**:

$$[\cdot,\cdot]\colon L \times L \to L,$$

  bilinear and skew-symmetric as a function of objects and morphisms,

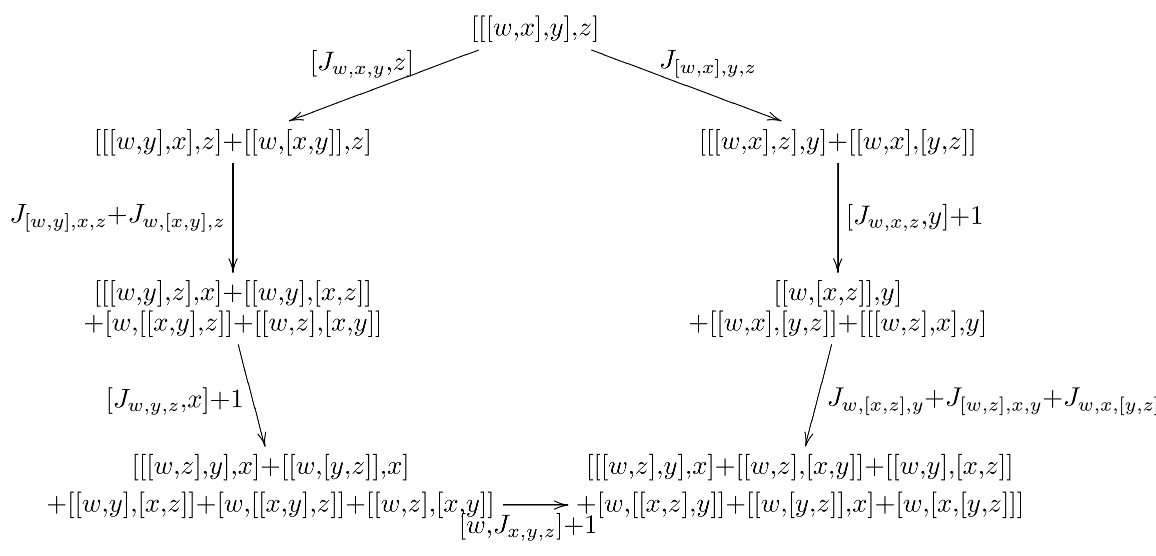
- a natural isomorphism called the **Jacobiator**:

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$

  trilinear and antisymmetric as a function of the objects x, y, z,

such that:

- the **Jacobiator identity** holds: the following diagram commutes:



We can also define homomorphisms between Lie 2-algebras, and 2-homomorphisms between these. So, the Lie 2-algebras L and L' are **equivalent** if there are homomorphisms

$$f\colon L \to L' \qquad \bar{f}\colon L' \to L$$

that are inverses up to 2-isomorphism.

Theorem. Lie 2-algebras are classified up to equivalence by quadruples consisting of:

- a Lie algebra $\mathfrak{g}$,
- an abelian Lie algebra (= vector space) $\mathfrak{h}$,
- a representation ρ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

Just like the classification of 2-groups, but with Lie algebra cohomology replacing group cohomology!

Let's use this to find some interesting Lie 2-algebras. Then let's try to find the corresponding Lie 2-groups. A **Lie 2-group** is a 2-group where everything in sight is smooth.

The Lie 2-Algebra $\mathfrak{g}_k$

Suppose $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{R}$. To get a Lie 2-algebra with $\mathfrak{g}$ as objects we need:

- a vector space $\mathfrak{h}$,
- a representation ρ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$.

Assume without loss of generality that ρ is irreducible. To get Lie 2-algebras with nontrivial Jacobiator, we need $H^3(\mathfrak{g}, \mathfrak{h}) \neq 0$. This only happens when $\mathfrak{h} = \mathbb{R}$ is the trivial representation. Then we have

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

with a nontrivial 3-cocycle given by:

$$\nu(x, y, z) = \langle [x, y], z \rangle.$$

Using k times this to define the Jacobiator, we get a Lie 2-algebra we call $\mathfrak{g}_k$.

In short: *every simple Lie algebra* $\mathfrak{g}$ *admits a canonical one-parameter deformation* $\mathfrak{g}_k$ *in the world of Lie 2-algebras!*

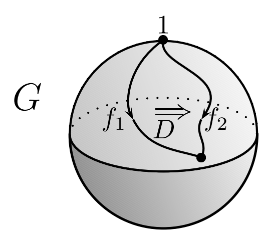
Does $\mathfrak{g}_k$ Come From a Lie 2-Group?

The bad news: while there is a 2-group that is 'trying' to have $\mathfrak{g}_k$ as its Lie algebra, it cannot be made into a Lie 2-group. It has G as its set of objects and $\mathrm{U}(1)$ as the endomorphisms of any object, but unless $k = 0$ we cannot make its associator everywhere smooth — only in a neighborhood of the identity!

But all is not lost. $\mathfrak{g}_k$ is *equivalent* to a Lie 2-algebra that *does* come from a Lie 2-group! However, this Lie 2-algebra is *infinite-dimensional!*

Theorem. For any $k \in \mathbb{Z}$, there is an infinite-dimensional Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra is *equivalent* to $\mathfrak{g}_k$.

An object of $\mathcal{P}_k G$ is a smooth path in G starting at the identity. A morphism from f_1 to f_2 is an equivalence class of pairs (D, α) consisting of a smooth homotopy D from f_1 to f_2 together with $\alpha \in \mathrm{U}(1)$:



There's an easy way to compose morphisms in $\mathcal{P}_k G$, and the resulting category inherits a Lie 2-group structure from the Lie group structure of G.

The Role of Loop Groups

We can also describe $\mathcal{P}_k G$ using central extensions of the loop group of G:

Theorem. An object of $\mathcal{P}_k G$ is a smooth path in G starting at the identity. Given objects $f_1, f_2 \in \mathcal{P}_k G$, a morphism

$$\widehat{\ell} \colon f_1 \to f_2$$

is an element $\widehat{\ell} \in \widehat{\Omega_k G}$ with

$$p(\widehat{\ell}) = f_2/f_1 \in \Omega G$$

where $\widehat{\Omega_k G}$ is the level-k central extension of the loop group ΩG:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \xrightarrow{p} \Omega G \longrightarrow 1$$

Since central extensions of loop groups play a basic role in string theory, and higher gauge theory is all about parallel transport of strings, this suggests $\mathcal{P}_k G$ is an interesting Lie 2-group!

An Application to Topology

Any simply-connected compact simple Lie group G has

$$\pi_3(G) = \mathbb{Z}.$$

There is a topological group $\widehat{G}$ obtained by killing the third homotopy group of G. When $G = \mathrm{Spin}(n)$, $\widehat{G}$ is called $\mathrm{String}(n)$.

Theorem. For any $k \in \mathbb{Z}$, the geometric realization of the nerve of $\mathcal{P}_k G$ is a topological group $|\mathcal{P}_k G|$. When $k = \pm 1$,

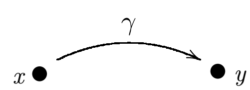
$$|\mathcal{P}_k G| \simeq \widehat{G}.$$

The group $\mathrm{String}(n)$ *shows up in string theory, especially elliptic cohomology — so this again suggests we're on the right track!*

Gauge Theory Revisited

Any manifold M gives a smooth groupoid $\mathcal{P}_1(M)$, its **path groupoid**, for which:

- objects are points $x \in M$: $\bullet_x$
- morphisms are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ that are constant near $t = 0, 1$:



For any Lie group G, a principal G-bundle $P \to M$ gives a smooth groupoid $\mathrm{Trans}(P)$, the **transport groupoid**, for which:

- objects are the fibers P_x (which are G-torsors),
- morphisms are G-torsor morphisms $f\colon P_x \to P_y$.

Via parallel transport, any connection on P gives a smooth functor called its **holonomy**:

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to \mathrm{Trans}(P)$$

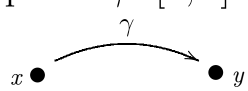
A trivialization of the bundle P makes $\mathrm{Trans}(P)$ equivalent to G, so we get:

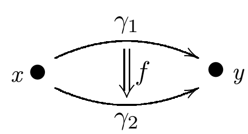
$$\mathrm{hol}\colon \mathcal{P}_1(M) \to G.$$

Higher Gauge Theory Revisited

We can categorify all the above and get a theory of *2-connections on principal 2-bundles.* See the papers by Toby Bartels and Urs Schreiber for details... or come with me to Canberra! With suitable definitions, it turns out that:

Any manifold M gives a smooth 2-groupoid $\mathcal{P}_2(M)$, its **path 2-groupoid**, for which:

- objects are points of M: $\bullet\, x$
- morphisms are smooth paths $\gamma\colon [0,1] \to M$ that are constant near $t = 0, 1$: 
- 2-morphisms are thin homotopy classes of smooth maps $f\colon [0,1]^2 \to M$ such that $f(s,t)$ is independent of s in a neighborhood of $s = 0$ and $s = 1$, and constant in a neighborhood of $t = 0$ and $t = 1$:



For any strict Lie 2-group $\mathcal{G}$, a principal $\mathcal{G}$-2-bundle $P \to M$ gives a smooth 2-groupoid $\mathrm{Trans}(P)$, the **transport 2-groupoid**, for which:

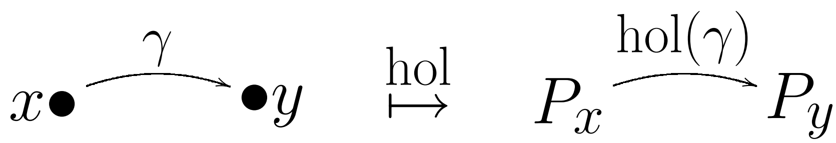
- objects are the fibers P_x (which are $\mathcal{G}$-2-torsors),
- morphisms are 2-torsor morphisms $f\colon P_x \to P_y$,
- 2-morphisms are 2-torsor 2-morphisms $\theta\colon f \Rightarrow g$.

Theorem. Via parallel transport, a 2-connection on P gives a smooth 2-functor called its **holonomy**:

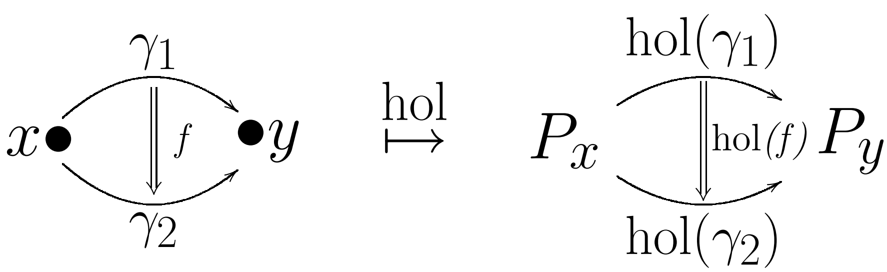
$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathrm{Trans}(P)$$

if and only if its 'fake curvature' vanishes.

So, in this case we can define the holonomy of our 2-connection along paths:



and paths-of-paths:



in a manner compatible with all 2-groupoid operations!

A trivialization of P makes $\mathrm{Trans}(P)$ equivalent to $\mathcal{G}$, so we get:

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}.$$

What Next?

1. Classify the representations of Lie 2-algebras and Lie 2-groups, especially $\mathfrak{g}_k$ and $\mathcal{P}_k G$.
2. Develop more categorified differential geometry: 2-bundles over smooth categories, the tangent 2-bundle of a smooth category, classifying 2-spaces of Lie 2-groups, and so on....
3. Develop physical theories based on 2-connections on 2-bundles — *higher gauge theories*.
4. Relate higher gauge theories to string theory and elliptic cohomology.
5. Go even higher: M-theory wants *3-connections on 3-bundles*, describing parallel transport of 2-branes. Read Urs Schreiber's thesis!