Spans and the Categorified Heisenberg Algebra – 3

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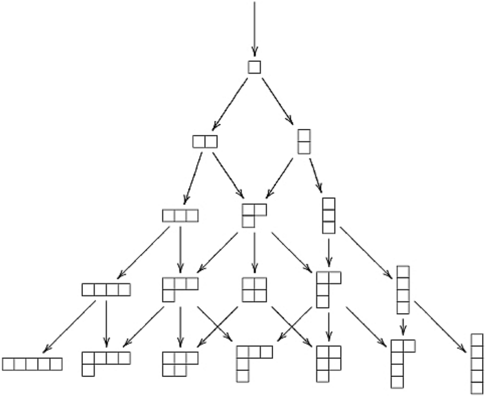


for more, see:

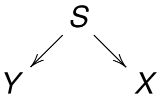
http://math.ucr.edu/home/baez/spans/

I'll start with a solid theorem:

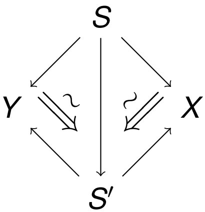
Theorem (Alex Hoffnung and Mike Stay)

There is a symmetric monoidal bicategory with:

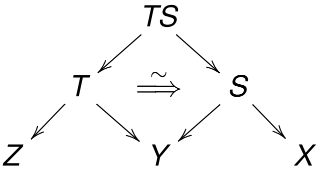
- *groupoids as objects*
- *spans of groupoids as morphisms:*



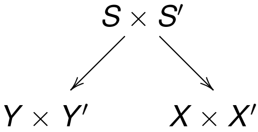
- *maps of spans as 2-morphisms:*



We compose spans by weak pullback:



and compose maps of spans in the obvious way. We tensor spans using products:



and similarly for spans of spans.

Morton and Vicary extrapolate this as follows:

Conjecture

There is a symmetric monoidal bicategory $\mathrm{Span}(\mathrm{Gpd})$ *with:*

- *groupoids as objects*
- *spans of groupoids as morphisms*
- *spans of spans as 2-morphisms:*

$$
\begin{array}{ccccc}
 & & S & & \\
 & \swarrow & \uparrow & \searrow & \\
Y & \Downarrow\wr & Z & \Downarrow\wr & X \\
 & \nwarrow & \downarrow & \nearrow & \\
 & & S' & &
\end{array}
$$

where we compose spans of spans using weak pullback.

To connect their theory of annihilation and creation operators to the work of Khovanov, they use representation theory.

A Kapranov–Voevodsky **2-vector space** is a $\mathbb{C}$-linear abelian category which is **semisimple**, meaning that every object is a finite direct sum of **simple** objects: objects that do not have any nontrivial subobjects.

Example

The category $\mathrm{FinRep}(G)$ of finite-dimensional (complex) representations of a group is $\mathbb{C}$-linear and abelian. The simple objects are the irreducible representations. If G is finite, $\mathrm{FinRep}(G)$ is a 2-vector space.

We can generalize this example to groupoids.

There is a category Vect of vector spaces and linear operators. A **representation** of a groupoid G is a functor $F\colon G \to \mathrm{Vect}$. A **morphism** of representations is a natural transformation $\alpha\colon F \Rightarrow F'$ between such functors.

Example

A groupoid G with one object can be seen as a group. A representation of G is the same as a representation of this group. Morphisms between representations are also the same as usual.

Say a representation of a groupoid G is **finite** if $F(x)$ is finite-dimensional for all objects $x \in G$, and zero-dimensional except for x in finitely many isomorphism classes.

Example

A groupoid G with one object can be seen as a group; then a finite representation of G is a finite-dimensional representation of this group.

Let $\mathrm{FinRep}(G)$ be the category of finite representations of the groupoid G.

Example

Let **S** be the groupoid of finite sets and bijections. **S** is equivalent to the coproduct

$$\sum_{n=0}^{\infty} S_n$$

where S_n is the symmetric group on n letters, seen as a one-object groupoid.

Thus, a representation $F\colon \mathbf{S} \to \mathrm{Vect}$ is the same as a representation F_n of S_n for each $n \geq 0$. F is finite and only if each F_n is finite dimensional and only finitely many are nonzero.

It follows that $\mathrm{FinRep}(\mathbf{S})$ is a 2-vector space.

More generally, say a groupoid is **locally finite** if all its homsets are finite. G is locally finite if and only if it is equivalent to a coproduct of finite groups. In this case $\mathrm{FinRep}(G)$ is a 2-vector space.

Let $\mathrm{Span}(\mathrm{FinGpd})$ be the symmetric monoidal bicategory with:

- locally finite groupoids as objects
- spans as morphisms
- spans of spans as 2-morphisms.

Conjecture

There is a symmetric monoidal functor

$$\mathrm{FinRep}\colon \mathrm{Span}(\mathrm{FinGpd}) \to 2\mathrm{Vect}$$

sending any locally finite groupoid G to its category of finite representations.

A few remarks on how the proof should go:

We make $2\mathrm{Vect}$ into a bicategory in the usual way, with:

- exact $\mathbb{C}$-linear functors as morphisms
- natural transformations as 2-morphisms.

We give it the usual tensor product, so that

$$\mathrm{FinVect}^m \otimes \mathrm{FinVect}^n \simeq \mathrm{FinVect}^{mn}$$

where $\mathrm{FinVect}$ is the category of finite-dimensional vector spaces, and more generally

$$\mathrm{FinRep}(G) \otimes \mathrm{FinRep}(H) \simeq \mathrm{FinRep}(G \times H)$$

How does

$$\mathrm{FinRep}\colon \mathrm{Span}(\mathrm{FinGpd}) \to 2\mathrm{Vect}$$

send a span of groupoids

$$\begin{array}{ccccc} & & S & & \\ & {}^{q}\swarrow & & \searrow^{p} & \\ Y & & & & X \end{array}$$

to an exact functor from $\mathrm{FinRep}(X)$ to $\mathrm{FinRep}(Y)$?

This was developed in a 2008 paper by Morton.

Given a functor $p\colon X \to Y$ between groupoids, we get

$$\begin{array}{rcl} p^*\colon \mathrm{FinRep}(Y) & \to & \mathrm{FinRep}(X) \\ F & \mapsto & F \circ p \end{array}$$

which has a left (and right!) adjoint

$$p_*\colon \mathrm{FinRep}(X) \to \mathrm{FinRep}(Y)$$

Both these are exact.

In group theory, p^* and p_* are called **restricting** and **inducing** representations along the homomorphism p. The fact that they're adjoint is called **Frobenius reciprocity**.

So, given a span of groupoids

$$\begin{array}{ccc} & S & \\ {}^{q}\swarrow & & \searrow^{p} \\ Y & & X \end{array}$$

we get an exact functor

$$\begin{array}{ccc} & \mathrm{FinRep}(S) & \\ {}^{q_*}\swarrow & & \nwarrow^{p^*} \\ \mathrm{FinRep}(Y) & & \mathrm{FinRep}(X) \end{array}$$

and Morton showed this sends composite spans to composite functors (up to natural isomorphism).

Using

$$\mathrm{FinRep}\colon \mathrm{Span}(\mathrm{FinGpd}) \to 2\mathrm{Vect}$$

we can map the whole theory of annihilation and creation operators into $2\mathrm{Vect}$!

In particular,

$$\textbf{Schur} = \mathrm{FinRep}(\textbf{S})$$

is called the category of **Schur functors**. It has one simple object for each Young diagram. It plays a basic role in representation theory, since it acts on the category of finite representations of any group, or groupoid:

$$\alpha\colon \textbf{Schur} \otimes \mathrm{FinRep}(G) \to \mathrm{FinRep}(G)$$

The annihilation operator

$$\begin{array}{ccc} & \mathbf{S} & \\ {}^{1_{\mathbf{S}}}\swarrow & & \searrow^{+1} \\ \mathbf{S} & & \mathbf{S} \end{array}$$

and creation operator

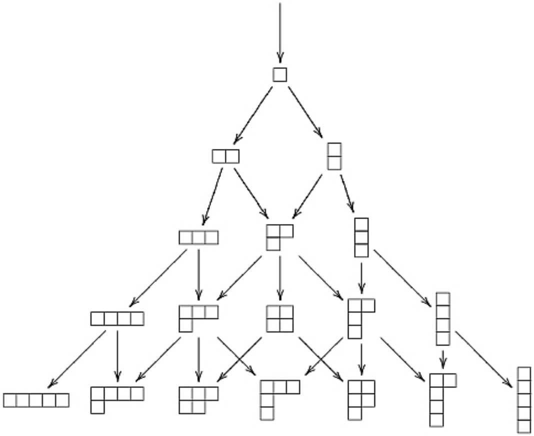
$$\begin{array}{ccc} & \mathbf{S} & \\ {}^{+1}\swarrow & & \searrow^{1_{\mathbf{S}}} \\ \mathbf{S} & & \mathbf{S} \end{array}$$

are spans from **S** to itself, so they should give exact functors

$$\mathbf{A} = \mathrm{FinRep}(A) \colon \mathbf{Schur} \to \mathbf{Schur}$$

$$\mathbf{A}^\dagger = \mathrm{FinRep}(A^\dagger) \colon \mathbf{Schur} \to \mathbf{Schur}$$

Morton and Vicary show $\mathbf{A}^\dagger : \mathbf{Schur} \to \mathbf{Schur}$ looks like this:



Remember that the canonical commutation relation

$$A^\dagger A + 1_{\mathbf{S}} \stackrel{\sim}{\Longrightarrow} AA^\dagger$$

gives two 'inclusions'

$$i\colon A^\dagger A \Rightarrow AA^\dagger \qquad j\colon 1_{\mathbf{S}} \Rightarrow AA^\dagger$$

such that i, j and their flipped versions $i^\dagger, j^\dagger$ are spans of spans obeying the relations in Khovanov's categorified Heisenberg algebra. FinRep should send all these to $2\mathrm{Vect}$:

$$\mathbf{i} = \mathrm{FinRep}(i) \qquad \mathbf{j} = \mathrm{FinRep}(j)$$

$$\mathbf{i}^\dagger = \mathrm{FinRep}(i^\dagger) \qquad \mathbf{j}^\dagger = \mathrm{FinRep}(j^\dagger)$$

where they should obey all the same relations.

Putting this all together, Morton and Vicary obtain:

The 2-vector space of Schur functors is equipped with exact functors

$$\mathbf{A}, \mathbf{A}^\dagger \colon \mathbf{Schur} \to \mathbf{Schur}$$

and natural transformations

$$\mathbf{i} \colon \mathbf{A}^\dagger\mathbf{A} \Rightarrow \mathbf{A}\mathbf{A}^\dagger \qquad \mathbf{j} \colon 1_{\mathbf{Schur}} \Rightarrow \mathbf{A}\mathbf{A}^\dagger$$

$$\mathbf{i}^\dagger \colon \mathbf{A}\mathbf{A}^\dagger \Rightarrow \mathbf{A}^\dagger\mathbf{A} \qquad \mathbf{j}^\dagger \colon \mathbf{A}\mathbf{A}^\dagger \Rightarrow 1_{\mathbf{Schur}}$$

obeying the relations in Khovanov's categorified Heisenberg algebra.

This is easy to check directly, but it would be nice to see it as part of a general theory!