EINSTEIN'S EQUATIONS: THE NEW VARIABLES

Canonical quantum gravity simplifies if for the kinematical phase space we use, not $T^*(\mathcal{M}et(S)) \ni (q,p)$:

S: compact oriented 3-manifold - "space"

$$q_{ij} = \text{metric on } S$$

$$p^{ij} = \sqrt{\det(q)}\,(K^{ij} - Kq^{ij})$$

extrinsic curvature of S

but instead $T^*\mathcal{A}$, where $\mathcal{A}$ is the space of smooth connections on the spin bundle $P \to S$ (a principal $SU(2)$ bundle).

γ Barbero-Immirzi parameter

$\frac{1}{\gamma}$

Meaning of the new variables $(A, E) \in T^*\mathcal{A}$:

- $A \in \mathcal{A}$ is a connection on P given by:

$$A = \Gamma - K$$

Γ: spin connection associated to metric q

K: extrinsic curvature as $Ad(P)$-valued 1-form:

$$K_i^a = e^{ja} K_{ij}$$

i, j : tangent space

a, b : $Ad(P)$ or "$\mathfrak{su}(2)$"

- $E \in T_A^*\mathcal{A}$ is a "densitized frame field":

$$T_A\mathcal{A} \cong \Omega^1(S) \otimes Ad(P)$$

$$T_A^*\mathcal{A} \cong Vect\underbrace{(S) \otimes Ad}_{\text{frame fields}}(P)^* \otimes \underbrace{\Omega^3(S)}_{\text{densities}}$$

Let $e \in Vect(S) \otimes Ad(P)$ have

$$e_a^i e_b^j = q^{ij} \delta_{ab}$$

and set

$$E = e \otimes vol$$

As a cotangent bundle, $T^*\mathcal{A}$ has the usual Poisson structure:

$$\{ E^i_a(x), A^b_j(y) \} = \delta^i_j \, \delta^b_a \, \delta(x,y)$$

and there are Poisson maps:

$$\begin{array}{ccc} T^*\mathcal{A} & & \\ \downarrow & & \\ T^*(\mathcal{A}/\mathcal{G}) & \xhookleftarrow[\text{open}]{\text{dense}} & T^*(\mathcal{M}et(S)) \end{array}$$

defined by Poisson reduction & relations between (A,E) & (q,p). Inverse

$$T^*(\mathcal{A}/\mathcal{G}) \xrightarrow{\text{partial!}} T^*(\mathcal{M}et(S))$$

defined only for $[(A,E)]$ such that

$$(\det q)\, q^{ij} = E^i_a \, E^j_b \, \delta^{ab}$$

gives <u>nondegenerate</u> q^{ij}.

To form the physical phase space, we do Poisson reduction thrice:

$$
\begin{tikzcd}
T^*\mathcal{A} \arrow[d, "\substack{\text{Gauss}\\ \text{constraint}}"'] & D_i E^i_a = 0 \\
T^*(\mathcal{A}/\mathcal{G}) \cong T^*(\mathcal{M}et(S)) \arrow[d, "\substack{\text{Diffeomorphism}\\ \text{constraint}}"'] & \\
T^*(\mathcal{A}/\mathrm{Aut}(P)) \arrow[d, "\substack{\text{Hamiltonian}\\ \text{constraint}}"'] & \mathrm{Aut}(P) = \mathcal{G} \rtimes \mathrm{Diff}_0(S) \\
X : \text{ physical phase space} &
\end{tikzcd}
$$

Now we get further with quantization:

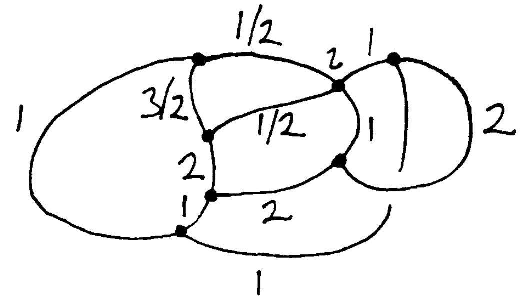
$$
\begin{tikzcd}[column sep=large, row sep=large]
T^*\mathcal{A} \arrow[r, "\text{QUANTIZE}"] \arrow[d, bend right, "\text{CONSTRAIN}"'] & L^2(\overline{\mathcal{A}}) \arrow[d, bend right, "\text{CONSTRAIN}"'] \\
T^*(\mathcal{A}/\mathcal{G}) \arrow[r, "\text{QUANTIZE}"] \arrow[d, bend right, "\text{CONSTRAIN}"'] & L^2(\overline{\mathcal{A}}/\overline{\mathcal{G}}) \arrow[d, bend right, "\text{CONSTRAIN}"'] \\
T^*(\mathcal{A}/\mathrm{Aut}(P)) \arrow[r, dotted, "\text{QUANTIZE}"] \arrow[d, bend right, "\text{CONSTRAIN}"'] & \mathcal{H}_{\mathrm{diff}} \arrow[d, dashed, "\text{CONSTRAIN}"'] \\
X \arrow[r, dotted, "\text{QUANTIZE}"] & \mathcal{H}_{\mathrm{phys}}
\end{tikzcd}
$$

$\longrightarrow$: solid $\quad$ $\dashrightarrow$: controversial

$\cdots\rightarrow$: still intractable

HILBERT SPACES FOR QUANTUM GRAVITY.

We've seen that the gauge-invariant Hilbert space $L^2(\bar{\mathcal{A}}/\bar{\mathcal{G}})$ has a basis of <u>spin networks</u>:



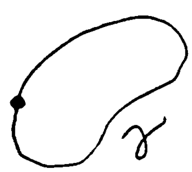
Similarly, $\mathcal{H}_{\mathrm{diff}}$ has a basis given by diffeomorphism equivalence classes of spin networks. But to understand the physical meaning of these spin network states, let's focus on $L^2(\bar{\mathcal{A}}/\bar{\mathcal{G}})$.

We need to introduce observables....

GAUGE-INVARIANT OBSERVABLES

On $L^2(\bar{\mathcal{A}}/\bar{\mathcal{G}})$, certain gauge-invariant functions of A act as multiplication operators. These capture information about parallel transport.

EXAMPLE: Wilson loops.



$$W(\gamma)(A) = \mathrm{tr}\left(\rho_{1/2}\left(e^{i\oint_\gamma A}\right)\right)$$

The trace of the holonomy of A around γ is a <u>Wilson loop</u>; multiplication by this function acts as a bounded self-adjoint operator on $L^2(\bar{\mathcal{A}}/\bar{\mathcal{G}})$.

Theorem: Finite linear combinations of spin networks form an algebra, generated by Wilson loops.

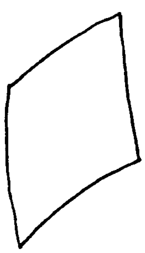
On $L^2(\bar{\mathcal{A}}/\bar{\mathcal{G}})$, certain gauge-invariant functions of E act as (pseudo) differential operators. Heuristically,

$$" \hat{E}^j_a(x) = \frac{1}{i} \frac{\partial}{\partial A^a_j(x)} "$$

These operators capture information about the metric.

EXAMPLE : Volume operators.

EXAMPLE : Area operators.



$\Sigma \subseteq S$

Classically, area of oriented surface Σ is:

$$A(\Sigma) = \gamma \int_\Sigma \sqrt{E \cdot E} := \gamma \int_\Sigma \sqrt{e^a e_a}\, \omega$$

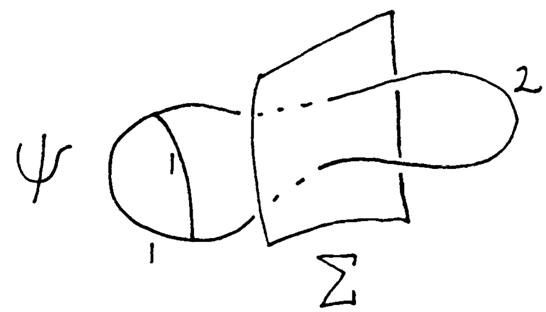
where $E = e\, \omega$

e: Ad(P)-valued function

ω: 2-form: $\mathrm{Vect}(S) \otimes_{C^\infty(M)} \Omega^3(S) \cong \Omega^2(S)$

Quantizing....

QUANTIZATION OF AREA



If ψ intersects Σ transversely, the area operator $\hat{A}(\Sigma)$ has

$$\hat{A}(\Sigma)\psi = \text{``}\gamma \int_{\Sigma} \sqrt{\hat{E}\cdot\hat{E}}\ \psi\text{''}$$

$$= \gamma \sum_{\substack{\text{edge } e \\ \text{punctures } \Sigma}} \sqrt{j_e(j_e+1)}\ \psi$$

So: spin network edges represent field lines of E field, & give area to surfaces they puncture. Minimal unit of area is

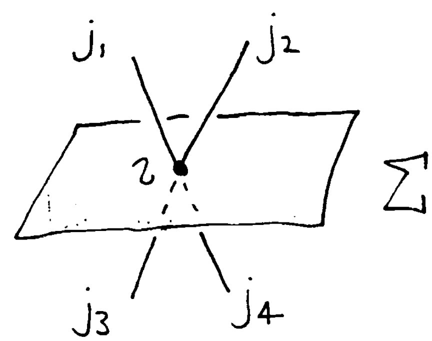
$$\frac{8\pi G\hbar}{c^3}\ \gamma\ \sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)} = \sqrt{3}\,\pi\,\gamma\,\ell_p^2$$

we'd been using units where this equals 1.

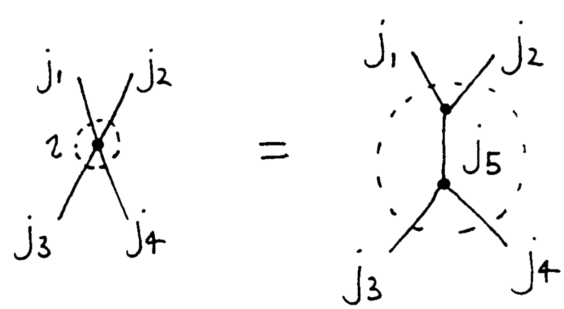
$\sim 3\cdot 10^{-70}$ meter^2

NONCOMMUTATIVITY OF AREA OPERATORS

Subtler phenomena occur in nongeneric cases, e.g.:



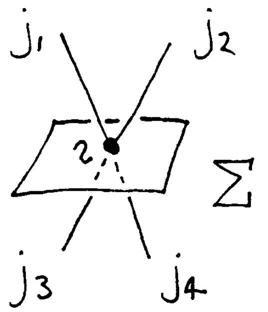
If



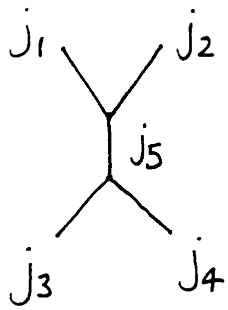
then

$$\hat{A}(\Sigma)\psi = \gamma \sqrt{j_5(j_5+1)}\ \psi$$

(back to units where $8\pi G = c = \hbar = 1$)

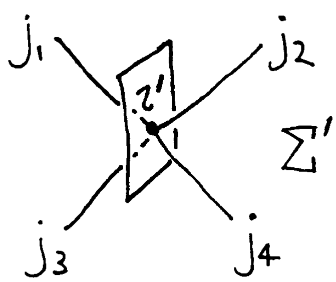


2 =

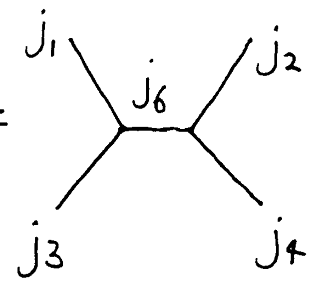


$$\Downarrow$$

$$\hat{A}(\Sigma)\psi = \gamma\sqrt{j_5(j_5+1)}\;\psi$$



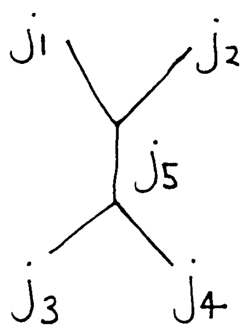
2' =



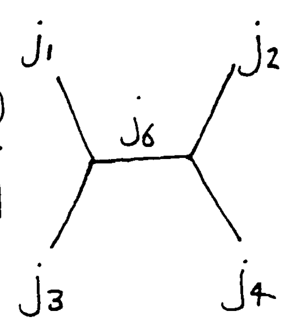
$$\Downarrow$$

$$\hat{A}(\Sigma')\psi = \gamma\sqrt{j_6(j_6+1)}\;\psi$$

Nontrivial change of basis:



$$= \sum_{j_6} \begin{Bmatrix} j_1 & j_2 & j_6 \\ j_4 & j_3 & j_5 \end{Bmatrix}$$



$$\Rightarrow \quad [\hat{A}(\Sigma), \hat{A}(\Sigma')] \neq 0 \, !$$