

6/2/03 Quarks & Standard Model

Gell-Mann saw that the 8 lightest mesons form a basis of

$$sl(3, \mathbb{C}) \cong su(3) \oplus \mathbb{C}$$

the complexified adj. rep of $SU(3)$.

What about other particles (hadrons) & other irreps of $SU(3)$?

What about the defining rep of $SU(3)$ on \mathbb{C}^3 ?

A 3-dim'l v. space so should have basis of 3 particles.

But not corresponding to anything seen! They have charges that are non-integral.

Gell-Mann called them "quarks"

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

up

$$d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

down

$$s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

strange

has isospin 0
in this theory
"eightfold way"

correspond to things w/
isospin up & down

Not true in current
theory.

$$I_3 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so, up & down are eigenvectors w/
eigenvalues $\pm 1/2$ - called
"isospin-up" and "isospin-down"

The strange quark shows up in "strange" particles -
not occurring in Heisenberg's $SU(2)$ theory i.e. Kaons.

The dual of the defining rep of $SU(3)$ is a rep on
 $(\mathbb{C}^3)^*$ (iso. as a vector space but not as rep).

This has a dual basis of "antiquarks":
row vectors

$$\bar{u} = (1 \ 0 \ 0) \quad \bar{d} = (0 \ 1 \ 0) \quad \bar{s} = (0 \ 0 \ 1)$$

We can use quarks q_i , antiquarks to build our
8 mesons, since:

$$\mathfrak{sl}(3, \mathbb{C}) \cong \mathbb{C}[3] \cong \mathbb{C}^3 \otimes (\mathbb{C}^3)^*$$

Given $f \in V^*$, $v \in V$, we get an elt $v \otimes f \in V \otimes V^*$
which gives a lin. transf. of V by:

$$(v \otimes f)(w) = f(w)v \in V \quad \forall w \in V.$$

v -quark, f -antiquark $v \otimes f$ - meson

Note - $\mathfrak{sl}(3, \mathbb{C})$ has 8 dims.

$\mathbb{C}[3]$ has 9 - is there another particle?

YES!

For example - $\pi^+ = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \approx \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes (0 \ 1 \ 0) = u \otimes \bar{d}$

In Heisenberg's theory, the positive pion was just upper left
 2×2 block.

So a positive pion is an up quark q_i anti-(down quark)

Particles are elements of a vector space, so it's
OKAY that some mesons are linear combs
of products of quark q_i , antiquark.

Like, for example: $\pi^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a "superposition" or "hybrid".

Mesons - are quark-anti-quark pairs

Baryons - built from 3 quarks.

2 quarks: $\mathbb{C}^3 \otimes \mathbb{C}^3 \cong S^2 \mathbb{C}^3 \oplus \Lambda^2 \mathbb{C}^3$ — $\dim = \frac{3 \cdot 2}{1 \cdot 2} = 3$

(the rep isn't reducible anymore! is direct sum of reps

$\dim = \frac{3 \cdot 4}{1 \cdot 2} = 6$

can pick same quark twice

can't pick same quark twice

$\dim \mathbb{C}^3 \otimes \mathbb{C}^3 = 9$, so $\mathbb{C}^3 \otimes \mathbb{C}^3$ is a sum of 2 irreps.

3 quarks $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \cong \mathbb{C}^3 \otimes (S^2 \mathbb{C}^3 \oplus \Lambda^2 \mathbb{C}^3)$

dim =

$\cong \underbrace{\mathbb{C}^3 \otimes S^2 \mathbb{C}^3}_{\dim 18} \oplus \underbrace{\mathbb{C}^3 \otimes \mathbb{C}^3}_{\dim 9}^*$ $\oplus \mathbb{C}^3^*$

$\cong S^3 \mathbb{C}^3 \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathbb{C}$

$\dim = \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} = 10$

$\dim = 8$

$\dim = 8$

$\dim = 1$
trivial rep

The more familiar baryons live in the "decuplet" $S^3 \mathbb{C}^3$ (gang of 10 things) & the "octet" $\mathfrak{sl}(3, \mathbb{C})$ (the 1st one shown).

$S^3 \mathbb{C}^3 =$ homogeneous deg-3 polynomials in u, d, s.

Ex) $\Omega^- = sss \in S^3 \mathbb{C}^3$

strange quark is heaviest,

So, Ω^- is the heaviest baryon in decuplet. Its properties were predicted by Gell-Mann before it was found, so won Nobel Prize.

6/3/03

Gell-Mann organized the hadron "zoo" into irreps of $SU(3)$ and postulated quarks as particles transforming in the defining rep of $SU(3)$ on \mathbb{C}^3 ; $SU(3)$ is symmetry group mixing up the u, d, s quarks.

Problem: The baryon decuplet, $S^3 \mathbb{C}^3$ contains particles like

$$\Lambda^- = sss \in S^3 \mathbb{C}^3$$

We have 3 identical quarks w/ same spin in the same state — violating Pauli exclusion principle.

Answer: postulate that quarks come in 3 "colors" — red, green & blue and that (flavors — up, down, strange)

$$\Lambda^- = S_r S_g S_b$$

Introduce a new $SU(3)$ that mixes up the 3 colors of quarks, in addition to Gell-Mann's $SU(3)$ mixing up the 3 "flavors" u, d, s . The whole symmetry group is then

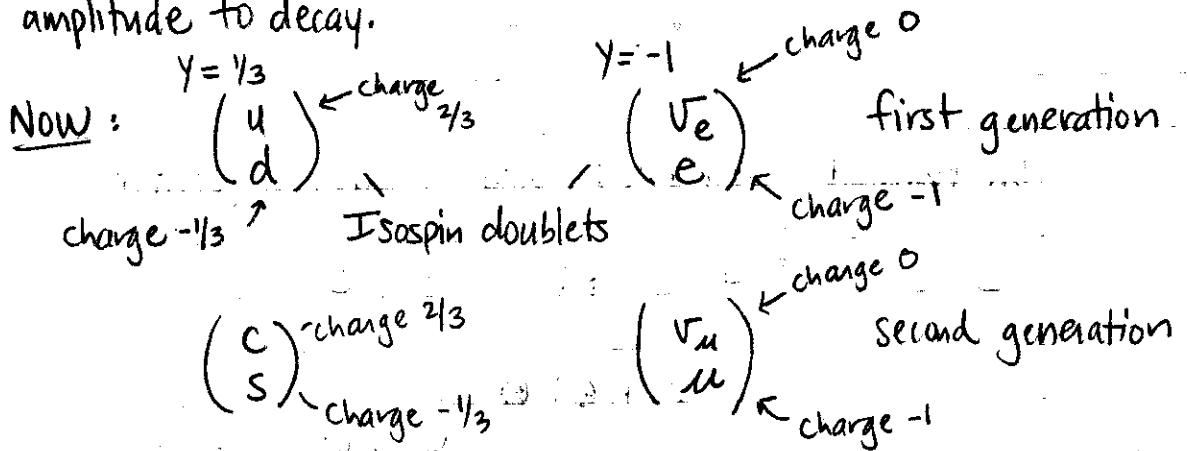
$$\underbrace{SU(3)}_{\text{flavor}} \times \underbrace{SU(3)}_{\text{color}}$$

approximate symmetry (eventually gets thrown out)

turns out to be fundamental — this is the one in the Standard Model $SU(3) \times SU(2) \times U(1)$.

Problem - Kaons decayed much slower than predicted.

Answer - Glashow - Iliopolis - Maiani suggested a fourth quark: "charm." This cancelled out some of the amplitude to decay.



u, d
produce
protons,
neutrons

$$Q = I_3 + \frac{Y}{2}$$

Salam also invented this extra quark just because it looked pretty. Mysterious - why is there one kind of lepton for each quark?

With 4 quarks, $SU(3)_{\text{flavor}}$ becomes $SU(4)_{\text{flavor}}$.
hadrons should be irreps of $SU(4)$.

Each new quark gives new mesons.

Mesons are now:

$$su(2) \otimes \mathbb{C} \subseteq su(3) \otimes \mathbb{C} \subseteq su(4) \otimes \mathbb{C}$$

ss

ss

ss

$$sl(2, \mathbb{C}) \subseteq sl(3, \mathbb{C}) \subseteq sl(4, \mathbb{C})$$

3-dim

8-dim

15-dim

So 15 dim. ↑

(since $sl(4, \mathbb{C})$ are 4×4 matrices (16 dim) w/ $\text{tr} = 0$)

$$sl(2, \mathbb{C}) : \begin{matrix} \pi^- & \pi^0 & \pi^+ \\ \cdot & \cdot & \cdot \end{matrix}$$

$$sl(3, \mathbb{C}) : \begin{matrix} & \cdot K^0 & \cdot K^+ \\ & & \cdot \pi^+ \\ \cdot \pi^- & \cdot \pi^0 & \\ & \cdot K^- & \cdot \bar{K}^0 \end{matrix}$$

On handout: $sl(4, \mathbb{C})$ has 4 guys in center.

$$\mathbb{C}^4 \otimes \mathbb{C}^{4*} = \mathbb{C}[4] \quad 16\text{-dim}$$

$$\cong sl(4, \mathbb{C}) \oplus \mathbb{C}$$

↳ multiples of id. matrix

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 0 & & & \\ 0 & & & \\ 0 & & & 0 \end{pmatrix} \otimes (1000) + \dots + \begin{pmatrix} 0 & & & \\ 0 & & & \\ 0 & & & \\ 0 & & & 1 \end{pmatrix} \otimes (0001)$$

$$= u \otimes \bar{u} + d \otimes \bar{d} + s \otimes \bar{s} + c \otimes \bar{c}$$

$$\dim \Lambda^p \mathbb{C}^n = \binom{n}{p}$$

Baryons now include:

$$\dim S^p \mathbb{C}^n = \binom{n+p-1}{p}$$

$$S^3 \mathbb{C}^2 \subseteq S^3 \mathbb{C}^3 \subseteq S^3 \mathbb{C}^4$$

$$\dim = \frac{\binom{2+3-1}{3}}{\binom{3+3-1}{3}} = \frac{2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3} = 4$$

$$\dim = \frac{\binom{3+3-1}{3}}{\binom{4+3-1}{3}} = \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} = 10$$

$$\dim = \frac{\binom{4+3-1}{3}}{\binom{5+3-1}{3}} = \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} = 20$$

But $SU(4)_{\text{flavor}}$ is even more approximate than $SU(3)_{\text{flavor}}$.

Eventually, we give up on $SU(n)_{\text{flavor}}$ and go to...

The Standard Model:

$$\underbrace{SU(3)}_{\text{mixes up colors of quarks}} \times \underbrace{SU(2)}_{\text{mixes up weak isospins e.g. } u \& d, \nu_e \& e} \times \underbrace{U(1)}_{\text{hypercharge}}$$

(Heisenberg - theory mixes up proton & neutron)
 (Gell-Mann - theory mixes up quarks, 3 things... but really amounts to mixing 2 things since
 proton = uud neutron = udd

Then we introduce force-carrying particles which live, as usual, in complexified adjoint rep of this group:

Strong Force - hold quarks together
 quarks stick together by transferring gluons

Strong force

$$(SU(3) \oplus SU(2) \oplus U(1)) \otimes \mathbb{C}$$

8 gluons - hold quarks together

W_1, W_2, W_3 or $W^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
 $W^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
 $W_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

W_0

The formula $Q = I_3 + \frac{Y}{2}$ gives

photon $\gamma = W_3 + \frac{W_0}{2}$ (not normalized)

The 2nd eqn on prev pg expresses the photon in terms of the W 's. But there's also

$$Z^0 = -\frac{1}{2}W_3 + W_0$$

chosen to be orthogonal to W^+ , W^- , and γ .

So we say $SU(2) \times U(1)$ is associated to the "electroweak force." We have

W^+, W^-	W bosons
Z^0	Z boson
γ	photon

$SU(2)$ w/ $[,]$ as observed particles.

same as

\mathbb{R}^3 w/

cross product

has no

preferred direction

Why is " I_3 direction" in $SU(2)$ special?

"Spontaneous symmetry breaking" due to Higgs boson. Let's look at some weak interactions.

$$\nu_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \quad e = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$$

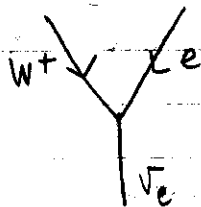
$$W^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C})$$

$$\mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

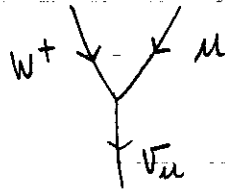
$$T \otimes v \longrightarrow Tv$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or}$$

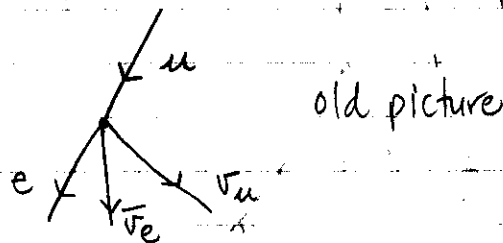
$$W^+ e \longrightarrow \nu_e$$



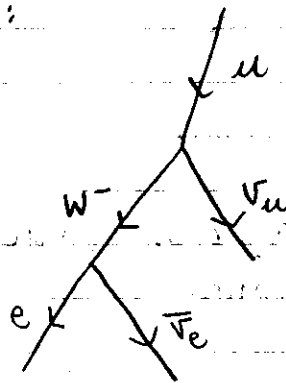
Also: same for 2nd generation:



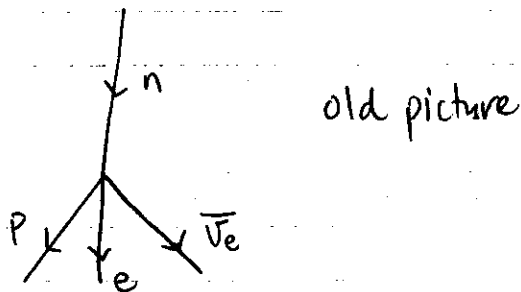
Sticking these together, we get muon decay:



is really:



Or - neutron decay:



But n, p are baryons - made up out of quarks.

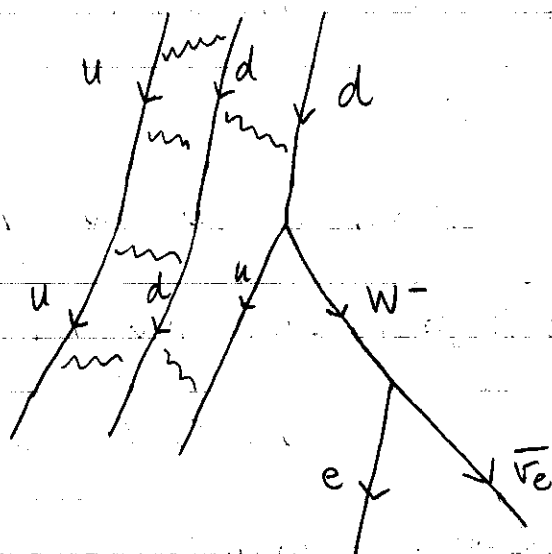
neutron
Really n is built from an up quark & 2 downs

$$+\frac{2}{3} - \frac{1}{3} - \frac{1}{3} = 0 \text{ charge}$$

p is built from a down & 2 ups

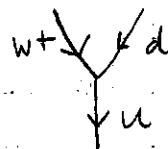
$$-\frac{1}{3} + \frac{2}{3} + \frac{2}{3} = 1$$

quarks
exchange
gluons &
colors



$$W^+ \otimes d \rightarrow u$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



End of Standard Model discussion!

Futuristic
Physics -
which may
not even
be true!

There's a subgroup of $SU(3) \times SU(2) \times U(1)$
called N , which acts as identity on all
particles in the Standard Model.

$$N \cong \mathbb{Z}_6$$

So - true symmetry group is

$$SU(3) \times SU(2) \times U(1) / N \cong S(U(3) \times U(2))$$

$$\cong \left\{ \begin{pmatrix} e^{-2i\theta} & 0 \\ 0 & e^{3i\theta} \end{pmatrix} \left| \begin{array}{l} g \in SU(3) \\ h \in SU(2) \\ e^{i\theta} \in U(1) \end{array} \right. \right\}$$

$$\subseteq SU(5)$$

Any rep of true gauge grp: $SU(3) \times SU(2) \times U(1)$ gives
a rep of quotient grp. (or vice versa)

We can try to lump together various irreps of $S(U(3) \times U(2))$
corresponding to different fermions, into irreps of $SU(5)$.
in a given generation

We'll think of reps of $SU(5)$ and see what reps of
 $S(U(3) \times U(2))$ they correspond to.

- 1) The trivial rep of $SU(5)$ on \mathbb{C} restricts to the
trivial rep of $S(U(3) \times U(2))$, or trivial rep of
 $SU(3) \times SU(2) \times U(1)$.

This rep corresponds to the right-handed electron
neutrino: $\mathbb{C}, \mathbb{C}, \mathbb{C}_0$.

(This was the last particle to be found — similar to
how zero is last natural # to be discovered,
since 'electron neutrino acts trivially.)

- 2) The defining rep of $SU(5)$ on \mathbb{C}^5 restricts to
 $\mathbb{C}^3 \oplus \mathbb{C}^2$ as a rep of $S(U(3) \times U(2))$ or as a
rep of $SU(3) \times SU(2) \times U(1)$:

defining

$$\underbrace{\mathbb{C}^3 \otimes \mathbb{C}^1 \otimes \mathbb{C}^1}_{d^R \text{ right-handed down quark}} \oplus \underbrace{\mathbb{C}^1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^1}_{\text{anti-}(\bar{\nu}_e^L, e^L) \text{ i.e. } (\bar{\nu}_e^R, \bar{e}^R)}$$

$\overset{\text{trivial}}{\mathbb{C}^1}$ $\overset{\text{triv.}}{\mathbb{C}^1}$ $\overset{\text{defining}}{\mathbb{C}^1}$
 $\overset{-2/3}{\mathbb{C}^3}$

$$\begin{pmatrix} e^{-2i\theta} & 0 \\ 0 & e^{3i\theta} \end{pmatrix}$$

if $\alpha = e^{i\theta}$ then $e^{-2i\theta}$ is how its acting

$$\alpha^{-2} = e^{-2i\theta} \quad \text{so } y = -2/3$$

if $\alpha = e^{i\theta}$, then acts as $e^{3i\theta}$
 $\alpha^3 = e^{3i\theta}$
 α^{3y} where $y=1$

Want a 10-dim'l rep of $SU(5)$.

3) The rep of $SU(5)$ on $\Lambda^2 \mathbb{C}^5$ has $\dim \frac{5 \cdot 4}{1 \cdot 2} = 10$,
 $(\mathbb{C} \oplus \mathbb{C}^5 \oplus \Lambda^2 \mathbb{C}^5)$

This is good because $1 + 5 + 10 = 16 = \dim F$
 where F is the "fermion rep".

Note: $\Lambda^2(V \oplus W) = \Lambda^2 V \oplus V \otimes W \oplus \Lambda^2 W$

gives

$$\begin{aligned} \Lambda^2 \mathbb{C}^5 &\cong \Lambda^2(\mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-2/3} \oplus \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_1) \\ &\cong \Lambda^2(\mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-2/3}) \oplus \underbrace{(\mathbb{C}^3 \otimes \mathbb{C}) \otimes (\mathbb{C} \otimes \mathbb{C}^2)}_{S// \mathbb{C}^3} \oplus \underbrace{(\mathbb{C} \otimes \mathbb{C}^2)}_{S// \mathbb{C}^2} \\ &\quad \oplus \Lambda^2(\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_1) \\ &\quad \oplus \underbrace{(\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}_{1/3})}_{S// \mathbb{C}_{1/3}} \\ &\cong \Lambda^2(\mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-2/3}) \oplus \underbrace{\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}_{1/3}}_{(u^+, d^+)} \\ &\quad \oplus \Lambda^2(\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_1) \end{aligned}$$

As reps of $SU(n)$, $\Lambda^p \mathbb{C}^n = \Lambda^{n-p}(\mathbb{C}^{n*})$ so

$$\Lambda^2 \mathbb{C}^3 = \Lambda^1 \mathbb{C}^{3*} = \mathbb{C}^{3*}$$

$$\cong \underbrace{\mathbb{C}^{3*} \otimes \mathbb{C} \otimes \mathbb{C}_{-4/3}}_{\text{anti-}(u^R) = \bar{u}^L} \oplus \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}_{1/3}$$

$$\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}_2 \rangle \text{ anti-}(e^R) = \bar{e}^L$$

$$\Lambda^2 \mathbb{C}^2 = \Lambda^0 \mathbb{C}^{2*} = \mathbb{C}$$

↑
0-forms - like functions

$$(\wedge^p \mathbb{C}^n)^* \cong \wedge^{n-p} \mathbb{C}^n$$

dual of p-form is
n-p form

Every fermion in F or its antiparticle shows up,
once, in

$$X = \mathbb{C} \oplus \mathbb{C}^5 \oplus \wedge^2 \mathbb{C}^5$$

so fermions ψ , their antiparticles are

$$F \oplus F^* = X \oplus X^* \quad \text{though } F \neq X$$

$$\begin{aligned} X \oplus X^* &= \mathbb{C} \oplus \mathbb{C}^5 \oplus \wedge^2 \mathbb{C}^5 \oplus \wedge^3 \mathbb{C}^5 \oplus \wedge^4 \mathbb{C}^5 \oplus \wedge^5 \mathbb{C}^5 \\ &\quad \underbrace{\quad}_{\wedge^0 \mathbb{C}^5} \quad \underbrace{\quad}_{\wedge^1 \mathbb{C}^5} \\ &= \wedge \mathbb{C}^5 \end{aligned}$$

Moral: Fermions ψ , antifermions form $\wedge \mathbb{C}^5$ as a rep of
 $S(U(3) \times U(2))$.