Session 2

Geometries & their algebraic groups

1) Elliptic Geometry.

Take $K^3 (R^3)$ with the usual inner product: $v \cdot w = \sum v_i w_i$.

From the sphere $X \subset \{ v \in K^3 : v \cdot v = 1 \}$

and define a set $P$ of points and $L$ of lines as follows:

$P = \{ 1d \text{ subspaces of } K^3 \}$

$L = \{ 2d \text{ subspaces of } K^3 \}$
and define the incidence relation: " point \( p \) lies on line \( l \) \( \iff \) \( p \in l \).

We use the dot product to define distances between points and angles between lines.

The group \( \text{SO}(3) \) acts on \( \mathbb{R}^3 \) preserving the dot product and also the incidence relation.

\[ p \in l \Rightarrow g(p) \in g(l), \quad \forall g \in \text{SO}(3) \]

2) Hyperbolic Geometry

Give \( \mathbb{R}^3 \) the "Lorentzian dot product"

\[ V \cdot W = -V_1W_1 - V_2W_2 + V_3W_3 \]

So that:

\[ X = \{ V \in \mathbb{R}^3 : V \cdot V < 1 \} \]
and

\( P = \{ 12 \text{ subspaces of } k^3 \text{ with non-empty intersection with } X \} \)

\[ L = \{ e_d \} \quad d = 2, 3, \ldots, 9 \]

Let

\[ O(1,2) = \{ g \in GL(3) : g \cdot v \cdot g^{-1} = v \cdot w \} \quad (\text{lorentzian dot product}) \]

\[ V, W \in k^3 \]

\( SO(1,2) \) preserves the dot product and incidence relation.

3) Euclidean Geometry

Take \( k^3 \) with the degenerate dot product:

\[ V \cdot N = V \cdot W \]

\[ V, N, W \in k^3 \]
To compare:

- **Elliptic:**
  - Dot product
  - $++$

- **Euclidean:**
  - $00+$

- **Hyperbolic:**
  - $-+$

Now

$X_s \{ \text{vek}^3 : vv=1 \}$

$P \{ \text{all subspaces of} k^3 \text{ having non-empty intersection} \}$

$L, \{ \text{2-d} \}$

What about the symmetry group?

The Euclidean group:

$$E(3) = \left\{ \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} : R \in SO(3) \text{ and } v \in k^2 \right\}$$
which preserves $X$:

\[
\begin{pmatrix}
R & V \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
R(x) + V \\
1
\end{pmatrix}
\]

[can we write $E(3)$ as a quotient group?]

and the distances and angles.

So the main weird difference with this case in that $E(3)$ is not the group of all $\text{det} = 1$ transformations preserving the degenerate dot product.

And Now:

4) Projective Geometry
Points and Lines:

\( P \subset \{12 \text{ subspaces of } k^3 \} \)

\( L \subset \{ 2d \} \)

We can't define distances and angles, but we can define the incidence relation: \( p \in \ell \).

Now the symmetry group is \( GL(3) \).

*Projective geometry goes back at least to the Renaissance painters, who developed perspective.*

If we fix any plane \( X \subseteq k^3 \) not containing the origin, most (but not all) 12 subspaces of \( k^3 \) will intersect \( X \) in a single point.
Thus most points $p \in \mathbb{P}$ correspond to points of $X$.

The remaining points of $\mathbb{P}$ are called "points at infinity".

Doing this for any dimension, we get projective $(n-1)$-space.

$k\mathbb{P}^{n-1} = \{1\text{d subspaces of } k^n\}$

Most points in $k\mathbb{P}^{n-1}$ will intersect the plane (for example)

$X = \{(x_0, \ldots, x_{n+1}) : x_i \in k\}$ in a single point.

So we get a 1-1 correspondence between $X$ and an "open dense" (what is the topology? Zariski) set in $k\mathbb{P}^{n-1}$.

Thus points and lines in 3 other geometries are points and line in projective geometry, and their symmetry group is a subgroup of $\text{GL}(2)$.
1) Projective Geometry

Recall: for any field \( k \):

\[ kP^n = \{ \text{1d subspaces of } k^{n+1} \} \]

Thin. As sets:

\[ kP^n \cong k^n \oplus k^{n-1} \oplus \ldots \oplus k^0 \]

isomorphic

means disjoint union

These pieces \( k^n, k^{n-1}, \ldots \) are called Schubert cells

As this is a Schubert decomposition

Proof: Any 1d subspace of \( k^{n+1} \) has the form

\[ \rho = \langle (a_1, \ldots, a_{n+1}) \rangle \cap (a_n, \ldots, a_{n+1}) \neq 0 \]

where \( \langle \cdot \rangle \) means span.
If \( n_{n+1} \neq 0 \)

\[ p = \langle \left( \frac{x_{n+1}}{n_{n+1}}, \ldots, \frac{x_n}{n_n}, 1 \right) \rangle = \langle y_1, y_2, \ldots, y_{n+1} \rangle \]

a description which is unique.

So we get a bijection between \( k^n \) and the set of all 1d subspaces of \( k^{n+1} \) of the form \( \langle \langle \alpha_{n+1}, \ldots, \alpha_{n+1} \rangle \rangle \) with \( n_{n+1} \neq 0 \).

If \( n_{n+1} \) then \( p \) is really a 1d subspace of \( k^n \)

\[ k^n = \left\{ (x_n, \ldots, x_0) : x \in k^n \right\} \]. So we get the bijection:

\[ kP^n \cong k^n + kP^{n-1} \]

by induction:

\[ kP^n \cong k^n + k^{n-2} + \cdots + k^0 \]
Examples:

1) $k P = k^1 + k^0$

- line $l$
- point at $\infty$

\[ \xrightarrow{1-1} \text{correspondence between lines with slope} \neq \infty \text{ and point on line } l \equiv k^1 \]

- line with slope $= \infty \equiv k^0$

2) $\mathbb{RP}^1$ as a set is $\mathbb{R} + \{\infty\}$. But as a topological space it is $S^1$, the 1-point compactification of $\mathbb{R}$.

3) $\mathbb{CP}^1 = \mathbb{C} + \{\infty\}$. As a space this is $S^2$, or as a complex variety, the Riemann sphere.
4) $\mathbb{R}P^2$ is homeomorphic to $S^2 \setminus \{u \cup v\}$

or to a disc $D^2$ with $u \cup v$ on boundary.

or to

\[ \begin{align*}
\text{the interior } &\subseteq \mathbb{R}^2 \\
\text{homeo to } &\mathbb{R} (\text{line at } \infty) \\
\text{homeo to } &\mathbb{R}^0
\end{align*} \]

$\Rightarrow \mathbb{R}P^2 \cong \mathbb{R}^2 + \mathbb{R} + \{\infty\}$

$\mathbb{R}P^1$

5) Any finite field $k$ has $q$ elements where $q$ is a prime power:

$q = p^m$ where $p$ is prime and $m \in \mathbb{N}$.

Moreover all fields with $q$ elements are isomorphic, so we write $\mathbb{F}_q$ for "the" field with $q$ elements.
If \( p \) prime is just \( \mathbb{Z}/p\mathbb{Z} \) with usual +.

To get \( \mathbb{F}_p^m \) with \( n > 1 \), we can take \( \mathbb{F}_p \) & throw in all \( m \) roots of some degree-\( m \) polynomial that has no roots in \( \mathbb{F}_p \).

What's the cardinality of \( \mathbb{F}_q P^n \)?

\[
|\mathbb{F}_q P^n| = |\mathbb{F}_q^n + \mathbb{F}_q^{n-1} + \cdots + \mathbb{F}_q^0|
\]

\[
= |\mathbb{F}_q^n| + |\mathbb{F}_q^{n-1}| + \cdots
\]

\[
= q^n + q^{n-1} + \cdots + 1
\]

\[
= \frac{q^{n+1} - 1}{q - 1}
\]

which is called the \( q \)-integer: \([n+1]_q\) (it approaches \( n+1 \) as \( q \to 1 \)).
6) $\mathbb{F}_2 P^2$ is called the "Fano plane," the smallest projective plane. $|\mathbb{F}_2 P^2| = 2^2 + 2 + 1 = 7$

**Diagram:**

- $e_1 + e_3$
- $e_2 + e_3$
- $e_1 = (1,0,0)$
- $e_2 = (0,1,0)$
- $e_3 = (0,0,1)$

**Diagram 2:**

- $\langle e_1 + e_3 \rangle$
- $\langle e_2 + e_3 \rangle$
- $\langle e_1, e_3 \rangle$
- $\langle e_2, e_3 \rangle$
- $\langle e_1, e_2 \rangle$
- $\langle e_1, e_3 \rangle$
- $\langle e_2, e_3 \rangle$

**Text:**

In $\mathbb{F}_2^3$, what are the 1D subspaces? What are the lines in the following figure?

**Text 2:**

For any two points, there is a line.

For any two lines, they intersect at one point.

**Thm.** In any projective plane $kP^2$:

1) For any 2 distinct points $p$ and $p'$, there is a line through $p$ with $p', p \in k$.

2) For any 2 distinct lines $l$, $l'$, there is a point $p$ with $p \in l, p \in l'$. 

(13)
Proof

1) Given 2 distinct 1d subspaces of $k^3$, $p$ & $p'$, the subspace sum $p + p'$ is $\{v + v', v \in p, v' \in p'\}$ is 2-dim, so it's a line with $p, p' \subset k$ and it is unique since...

2) Given 2 distinct 2d subspaces $l, l' \subset k^3$ we claim $l \cap l'$ will be a line subspace $p \subset l \cap l'$, and the clearly $p \subset l, l'$.
Def. An abstract projective plane consists of a set of points \( P \), a set of lines \( L \), and an incidence relation \( I \subseteq P \times L \). If \((p, e) \in I\) we say \( p \text{ lies on } e \).

Now we demand:

1) For any distinct \( p, p' \in P \) \( \exists! \, e \in L \) \( p \text{ lies on } e, p' \text{ lies on } e \)

2) For any distinct \( e, e' \in L \) \( \exists! \, p \in P \) \( p \text{ lies on } e, p \text{ lies on } e' \)

3) Non-degeneracy axiom:

V1. There exist 4 points, no 3 of which lie on the same line.

V2. There exist 4 lines, no 2 of which contain the same point.

The two versions are equivalent.
The non-degeneracy eliminates:

1) \( P = L = \emptyset \)

2) 

3) 

4) 

5) 

6) 

7) 

with these axioms, do projective six planes come from a field?
Last time, "Schubert cell" should have been "Bruhat cell".

A Bruhat cell is isomorphic to $k^n$; the corresponding Schubert cell is the closure of the Bruhat cell.

So if $k=\mathbb{R}$, Bruhat cells are open $n$-balls & Schubert cells are closed $n$-balls.

**Projective Planes & Axiomatic Projective geometry**

"Around 400 AD, Pappus wrote about Euclid's "Porisms".

Thm. Pappus's Hexagon theorem: if $k$ is any field and we have this configuration of points & lines in $kP^2$,

then points $a$, $b$, and $c$ lie on a line.

Proof: See Wikipedia.
In fact:

Thus, an abstract projective plane is isomorphic to the plane $\mathbb{P}^2$ coming from some field iff Pappus' hexagon theorem holds in this abstract projective plane.

**Klein Geometry**

Any kind of (highly symmetrical) geometry corresponds to a group $G$ (the symmetry group of the geometry).

There will be various sorts of "figures" (e.g., points, lines, circles, triangles, ...) and $G$ acts on each of these sets. In fact, we demand each set is a "homogeneous space" for $G$.

**Def.** Action of group $G$ on a set $X$: $\alpha: G \times X \to X$

$$\alpha(g, x) = g \cdot x$$

obeying:

* Associative law $g(hx) = (gh) \cdot x$ \hspace{1cm} \forall g, h \in G, \forall x \in X$

* Identity law $1 \cdot x = x$ \hspace{1cm} \forall x \in X$
We call this an action of $G$ on $X$, or a $G$-set, or a $G$-space.

**Def.** An action of $G$ on $X$ is transitive if \( \forall y \in X, \exists g \in G : gx = y \)

A transitive $G$-set is called a homogeneous $G$-space.

**Example:** if $G = E(n)$ (the Euclidean group) the the set of points $P = \mathbb{R}^n$ in a homogeneous $G$-space, as is the set of lines $L$.

The same holds for points & lines in the other geometries we've discussed:

- elliptic $G = SO(3)$
- hyperbolic $G = SO(1,2)$
- projective $G = GL(n)$

**Thm.** If $G$ is a group & $X$ is a transitive $G$-set, for any $x \in X$

there is an isomorphism (a bijection) $\phi : G/G_x \rightarrow X$

where $G_x$ is the stabilizer of $x \in X$: $G_x = \{ g \in G : gx = x \}$.

and $\phi([g]) = gx$.
Proof: 1) \( \phi \) is well defined: if \([g'] = [g] \) \( (g'vgh \text{ for some } h \in G) \)

\[ \phi([g']) = \phi([g]) \] since \( gx, ghx = gx \)

2) and also 1-1: if \( g(x, g(x)) \Rightarrow [g'] = [g] \)

3) and also onto: given any \( k \in X \), transitivity implies \( \exists g \in G \) s.t. \( gx = x' \Rightarrow \phi([g]) = x' \)

Note: the empty set always can be made into a \( G \)-set in one way, and this action is transitive. But \( \phi \neq G/H \) (for some \( H \)).

So did the above theorem go wrong? No. Since \( \exists x \in X \)

Example: 1) Euclidean plane: \( G \cong E(2) \)

the subgroup \( H = \left\{ \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \right\} \)

stabilizes a point, namely

\[ p = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \]
Here $H = \text{SO}(2) \cdot \text{SO}$ in Euclidean geometry:

$$P = E(2)/\text{SO}(2) \sim \text{really } H$$

2) $G$, $E(2)$, translations along a line fix that line.

So let $H' = \langle \begin{pmatrix} \ast & \ast & \ast \\ \ast & 1 & \ast \\ \ast & \ast & 1 \end{pmatrix} \rangle$

It stabilizes:

$$\mathcal{L} = \langle (0), (1) \rangle$$

but so does the $180^\circ$ rotation $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So $G \leq \langle H', \text{ 180 } \text{ rotation} \rangle \Rightarrow E(2) \, \cong \, L \left[ \text{the space of all lines} \right]$ \frac{\text{by } G}{G}$

also $G/\langle H' \rangle$ would be the space of oriented lines.
In Klein geometry, we pick a group \( G \) and say each subgroup \( H \subseteq G \) determines a "type of figure," with \( G/H \) being the set of figures of that type.

We can define incidence relations between two types of figures to be \( G \)-invariant relations.

Given two types of figures: \( X = G/H \) and \( X' = G/H' \), a relation between these is a subset \( R \subseteq X \times X' \).

\( R \) is \( G \)-invariant if:

\[
(x, x') \in R \Rightarrow (gx, gx') \in R \quad \forall g \in G
\]

E.g., in Euclidean geometry, "point on line" defines an \( E(1) \)-invariant relation: \( I \subseteq P \times L \).

* Starting from a group, we can work out all its subgroups & all the invariant relations on the resulting transitive \( G \)-sets.
According to Klein, a group gives a geometry.

So let's try: \( G, \text{GL}(n) \)

Different types of geometrical figure in projective geometry correspond to different subgroups of \( \text{GL}(n) \).

Here are some fundamental kind of figures:

**Def.** Let the Grassmannian \( \text{Gr}(n,j) \) \((1 \leq j \leq n-1)\) be the set of all \( j \)-dimensional linear subspaces of \( k^n \).

**Examples:**
- \( \text{Gr}(n,1) = kP^{n-1} = \{ \text{points of } kP^{n-1} \} \)
- \( \text{Gr}(n,2) = \{ \text{lines in } kP^{n-1} \} \)
- \( \text{Gr}(n,3) = \{ \text{planes in } kP^{n-1} \} \)
- \( \text{Gr}(n,4) = \{ (j-1) \text{-planes in } kP^{n-1} \} \)
- \( \text{Gr}(n,n-1) = \{ \text{hyperplanes in } kP^{n-1} \} \)
GL(n) acts on each Gr(n,j) via:

\[ GL = \{ g \in L : v \in L \} \leq Gr(n,j) \]

and they all are homogeneous spaces of GL(n):

any \( L \leq Gr(n,j) \) has a basis \( v_1, \ldots, v_j \in k^n \) & similarly \( L' \leq Gr(n,j) \) has basis \( v'_1, \ldots, v'_j \in L' \)

and we can find \( g \in GL(n) \) s.t. \( g v_i = v'_i \) so that \( g L = L' \)

thus by our theorem last time \( Gr(n,j) = GL(n)/P_{nj} \)

where \( P_{nj} \) is the subgroup that fixes a chosen \( L \in Gr(n,j) \)

The subgroups \( P_{nj} \) are "maximal parabolic" subgroups of \( GL(n) \).

Indeed, any (linear) algebraic group \( G \) will have maximal parabolic subgroups that fix the "nicest" types of figures in its geometry.

To study \( P_{nj} \) choose a nice j-dim subspace of \( k^n \):

\[ L = \{ (x_1, \ldots, x_j, 0, 0, \ldots, 0) \in k^n \} \]
8. define:

\[ \text{Proj} = \{ g \in \text{GL}(3) : gL = L \} \]

What's it like?

Examples.

\[ P_{3,1} = \text{subgroup of } \text{GL}(3) \text{ that fixes a point in the projective plane.} \]

\[
\begin{pmatrix}
* & * & * \\
0 & * & * \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
* \\
* \\
0
\end{pmatrix}
= \begin{pmatrix}
* \\
* \\
0
\end{pmatrix}
\]

This matrix fixes \( L = P_{3,1} \).

Any vector \( v \in L \) looks like this, where \( \ast \) means an arbitrary element of the field.

\[ P_{3,2} = \text{subgroup of } \text{GL}(3) \text{ that fixes a line in the projective plane.} \]

\[
\begin{pmatrix}
* & * & * \\
* & * & * \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
* \\
* \\
* 
\end{pmatrix}
= \begin{pmatrix}
* \\
* \\
* 
\end{pmatrix}
\]

\[ = P_{3,2} \]

In fact, \( P_{3,2} \cong P_{3,1} \) as a group but they are not conjugate in \( \text{GL}(3) \).
\[ P_{n,1}: \]
\[
4-1 \left\{ \begin{pmatrix} x & x & \cdots & x \\ 0 & x & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \right) \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ \vdots \\ \vdots \\ 0 \end{pmatrix}
\]

\[ P_{n,2}: \]
\[
4-2 \left\{ \begin{pmatrix} x & x & \cdots & x \\ 0 & x & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \right) \begin{pmatrix} x \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ \vdots \\ \vdots \\ 0 \end{pmatrix}
\]

\[ P_{n,3} \]
\[
4-3 \left\{ \begin{pmatrix} x & x & \cdots & x \\ 0 & x & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \right) \begin{pmatrix} x \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ \vdots \\ \vdots \\ 0 \end{pmatrix}
\]

Theorem:
\[
P_{n,j} = \left\{ \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} : X, Y, Z \text{ are arbitrary matrices of the correct shape} \right\}
\]

\[ \text{if } k=\mathbb{R}, \text{ any linear algebraic group } G \text{ is a manifold, and if } H \subseteq G \text{ is algebraic as well then } G/H \text{ is also a manifold if } \dim(G/H) = \dim G - \dim H \]
For any field, an (linear) algebraic group $G$ is an (affine) algebraic variety, and if $H \subset G$ is algebraic as well, then $G/H$ is also an algebraic variety (not necessarily affine) and
\[ \dim(G/H) = \dim(G) - \dim(H) \]

**Thm.**
\[ \dim(\text{Gr}(n,j)) = j(n-j) \]
from the form of $P_{n,j}$
\[ \dim(P_{n,j}) = n^2 - (n-j)j \]
and since $\text{Gr}(n,j) \cong \frac{P_{n,j}}{P_{n,j}}$, we have
\[ \dim(\text{Gr}(n,j)) = n-j \]

Moreover, $\dim(\text{Gr}(n,j)) = \dim(\text{Gr}(n,n-j))$ and in fact $\text{Gr}(n,j) \subseteq \text{Gr}(n,n-j)$ since using an inner product we get a 1-1 & onto map
\[ L \rightarrow L^* \]
which is called duality in projective spaces.
a Pseudo-Pascal's triangle for $d_{n,j} := \dim (Gr(n,j))$

But this is just the multiplication table!!

\[
\begin{align*}
    d_{1,0} &= 1 \\
    d_{2,0} &= 2 \\
    d_{2,1} &= 2 \\
    d_{3,0} &= 3 \\
    d_{3,1} &= 4 \\
    d_{3,2} &= 3 \\
    d_{4,0} &= 4 \\
    d_{4,1} &= 6
\end{align*}
\]

*Pascal's triangle shows up when we count the number of points in $Gr(n,j)$ when $k = \mathbb{F}_q$. We'll get the "$q$-deformed" Pascal's triangle.*

Remember $kP^n = k^n + k^{n-1} + \cdots + k^0$

So if $k = \mathbb{F}_q$:

\[
|kP^n| = \frac{q^{n+1} - 1}{q-1} = [n+1]_q \quad \text{(called $n+1$-th $q$-integer)}
\]

Also: $kP^{n-1} = Gr(n,1)$. How does $|Gr(n,1)| = [n]_q$ generalize to other Grassmannians?
Def. The q-factorial $[n]_q!$ is given by:

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

and the q-binomial coefficient $\left(\begin{array}{c} n \\ j \end{array}\right)_q$ is given by:

$$\left(\begin{array}{c} n \\ j \end{array}\right)_q = \frac{[n]_q!}{[j]_q! [n-j]_q!}$$

Thm. if $k = \mathbb{F}_q$ then $|\text{Gr}(W, J)| = \left(\begin{array}{c} n \\ j \end{array}\right)_q$

Note: $\left(\begin{array}{c} n \\ j \end{array}\right)$ counts the number of j-element subsets of an n-element set, while $\left(\begin{array}{c} n \\ j \end{array}\right)_q$ counts the number of j-dimensional subspaces of a n-dimensional vector space over $\mathbb{F}_q$.

So in some mysterious way a vector space over "the field with an element" ($\mathbb{F}_q$) is just a finite set!"