1. Projective Geometry from a Kleinian Perspective

According to Klein, a linear algebraic group “gives” us a geometry. Let’s first consider \( G = GL(n) \). In general, different sets of figures in projective geometry correspond to different subgroups of \( GL(n) \).

**Definition.** Let the Grassmannian \( Gr(n, j) \), for \( 0 \leq j \leq n \), be the set of all \( j \)-dimensional subspaces of \( k^n \).

As examples, we have that:
\[
\begin{align*}
Gr(n, 1) &= \{ \text{points of } kP^{n-1} \}, \\
Gr(n, 2) &= \{ \text{lines of } kP^{n-1} \}, \\
Gr(n, 3) &= \{ \text{planes of } kP^{n-1} \}, \\
Gr(n, j) &= \{ (j-1)\text{-planes of } kP^{n-1} \}, \\
Gr(n, n-1) &= \{ \text{hyperplanes of } kP^{n-1} \}.
\end{align*}
\]

Now, \( GL(n) \) acts on each Grassmannian, since it acts on \( k^n \) mapping subspaces to subspaces of the same dimension. If \( L \in Gr(n, j) \) and \( g \in GL(n) \), then \(gL = \{ gv : v \in L \} \).

The Grassmannians are all homogeneous \( GL(n) \)-spaces, which is to say \( GL(n) \) acts transitively. Any \( L \in Gr(n, j) \) has a basis \( \{v_i\}_{i=1}^j \), and any other \( L' \in Gr(n, j) \) has a basis \( \{v_i'\}_{i=1}^j \). Basic linear algebra tells us we can find a linear operator \( g \in GL(n) \) such that \( gv_i = v_i' \) for all \( i \), so there exists a \( g \) such that \(gL = L' \).

By our “easy” theorem in the last lecture, a Grassmannian is in bijection with a quotient space \( Gr(n, j) \cong GL(n)/P_{n,j} \) for linear algebraic subgroup \( P_{n,j} \), where \( P_{n,j} \) is the subgroup that fixes a chosen \( L \in Gr(n, j) \). Subgroups of this form - \( P_{n,j} \) - are “maximal parabolic” subgroups of \( GL(n) \). Indeed, any linear algebraic group will have some maximal parabolic subgroups that fix the “nicest” types of figures in its associated geometry.

2. Maximal Parabolic Subgroups

To study these \( P_{n,j} \) choose a nice \( L \in kP^{n-1} \). We can choose \( L = \{ (x_1, x_2, \ldots, x_j, 0, 0, \ldots, 0) \in k^n \} \), and define \( P_{n,j} = \{ L \in GL(n) :gL = L \} \).

We begin with a few basic examples. First, we work in \( k^3 \).

*Date: October 10, 2016.*
Example 1. We can consider

\[ P_{3,1} = \{ g \in GL(3) : g \text{ fixes a point in the projective plane.} \} \]

We will use the convention that a star (\( \ast \)) is a wildcard, which can take any value in \( k \). This means our particular \( L \) can be written as \( ( \ast \ 0 \ 0 )^T \), and by testing the idea we get that the subgroup consists of matrices of the form

\[
\begin{pmatrix}
\ast & \ast & \ast \\
0 & \ast & \ast \\
0 & 0 & \ast
\end{pmatrix}
\begin{pmatrix}
\ast \\
0 \\
0
\end{pmatrix} =
\begin{pmatrix}
\ast \\
0 \\
0
\end{pmatrix}.
\]

Hence,

\[ P_{3,1} = \{ A \in GL(3) : a_{21}, a_{31} = 0 \} . \]

Similarly, we could choose a nice line in \( kP^{n-1} \) as things of the form \( ( \ast \ \ast \ 0 )^T \). Then, \( P_{3,2} \) would be things of the form

\[
\begin{pmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
0 & 0 & \ast
\end{pmatrix}
\begin{pmatrix}
\ast \\
0 \\
0
\end{pmatrix} =
\begin{pmatrix}
\ast \\
0 \\
0
\end{pmatrix}.
\]

Thus,

\[ P_{3,2} = \{ A \in GL(3) : a_{31}, a_{32} = 0 \} . \]

The cases in \( k^3 \) aren’t very illuminating, aside from showing that \( P_{3,1} \cong P_{3,2} \) as a group. However, they are not conjugate, which is to say there is no \( g \in GL(3) \) such that \( gP_{3,1}g^{-1} = P_{3,2} \).

Let’s briefly look at a slightly larger case.

Example 2. Now we will work in \( k^4 \). Through the process shown above, we have

\[
\begin{pmatrix}
\ast & \ast & \ast & \ast \\
0 & \ast & \ast & \ast \\
0 & \ast & \ast & \ast \\
0 & \ast & \ast & \ast
\end{pmatrix} \in P_{4,1}, \quad \begin{pmatrix}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
0 & 0 & \ast & \ast \\
0 & 0 & \ast & \ast
\end{pmatrix} \in P_{4,2},
\]

and finally

\[
\begin{pmatrix}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
0 & 0 & 0 & \ast
\end{pmatrix} \in P_{4,3}.
\]

Notice that in each case, we replace the lower left corner by a zero matrix with \( n - j \) rows and \( j \) columns. This leads to the general result.

Theorem. For a field \( k \),

\[ P_{n,j} = \left\{ \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} : X \in GL(j), Z \in GL(n-j) \right\} . \]

This clearly means that \( P_{n,j} \) is a subgroup of \( GL(n) \).
3. Consequences

If \( k = \mathbb{R} \), any linear algebraic group is a manifold, so we can speak of its dimension. If \( H \) is a linear algebraic subgroup of \( G \), then \( G/H \) is also a manifold, and

\[
\dim(G/H) = \dim G - \dim H.
\]

However, for an arbitrary field \( k \), a linear algebraic group need not be a manifold, but is instead called an (affine) algebraic variety. If \( H \subseteq G \) is also a linear algebraic group, then \( G/H \) is an algebraic variety as well - but not affine. Like the case for \( \mathbb{R} \), however, we can find the dimension as

\[
\dim(G/H) = \dim G - \dim H.
\]

From this, we can show that

**Theorem.** The dimension of a Grassmannian is given by

\[
\dim(Gr(n, j)) = j(n - j).
\]

**Proof.** Recall that elements of \( P_{n,j} \) are matrices in \( GL_n \) which have an \( n - j \) by \( j \) zero matrix in the lower left corner, so

\[
\dim P_{n,j} = n^2 - (n - j)j.
\]

Utilizing our isomorphism,

\[
\dim(Gr(n, j)) = \dim(\frac{GL(n)}{P_{n,j}}) = \dim GL(n) - \dim P_{n,j} = n^2 - (n^2 - (n - j)j) = (n - j)j.
\]

\( \square \)

When we look at dimensions of Grassmannians, there’s something akin to Pascal’s triangle lurking about. If we construct something of the form

\[
\begin{array}{cccccccc}
\dim(Gr(0,0)) & \dim(Gr(1,0)) & \dim(Gr(1,1)) & \dim(Gr(2,0)) & \dim(Gr(2,1)) & \dim(Gr(2,2)) & \dim(Gr(3,0)) & \dim(Gr(3,1)) & \dim(Gr(3,2)) & \dim(Gr(3,3)) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
0 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\
0 & 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\
0 & 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\
\end{array}
\]

this numerically looks like

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
0 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\
0 & 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\
0 & 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\
\end{array}
\]

which is a rotated version of the multiplication table!
However, a Pascal’s triangle also shows up. If \( k = \mathbb{F}_q \), where \( q = p^n \) for some prime \( p \), we get what is known as a \( q \)-deformed Pascal’s triangle. Recall the decomposition of projective space into Bruhat cells:

\[
kP^n \cong k^n + k^{n-1} + \cdots + k^0.
\]

If \( k = \mathbb{F}_q \), then

\[
|kP^n| = |k^n| + |k^{n-1}| + \cdots + |k^0| = q^n + q^{n-1} + \cdots + 1 = \frac{q^{n+1} - 1}{q - 1} = [n + 1]_q,
\]

the \( q \)-integer we defined previously. But we already know that \( kP^{n-1} = Gr(n, 1) \). Thus \( |Gr(n, 1)| = [n]_q \).

How does this fact generalize?

**Definition.** The \( q \)-factorial \([n]_q!\) is given by

\[
[n]_q! = [n]_q \cdot [n - 1]_q \cdots [1]_q,
\]

and the \( q \)-binomial coefficient is given by

\[
\binom{n}{j}_q = \frac{[n]_q!}{[j]_q! \cdot [n - j]_q!}.
\]

We will finally state without proof (until the next lecture),

**Theorem.** If \( k = \mathbb{F}_q \), then

\[
\dim (Gr(n, j)) = \binom{n}{j}_q.
\]

Note the interesting analogy: \( \binom{n}{j} \) counts the number of \( j \)-element subsets of a set of size \( n \), while \( \binom{n}{j}_q \) counts the number of \( j \)-dimensional subspaces in \( \mathbb{F}_q^n \).

In some mysterious sense, a vector space over the “field with one element” (so \( q = 1 \)) is just a finite set!