1. Bruhat Cells

Considering an arbitrary field $k$, recall that

$$kP^n := \{1\text{-dimensional subspaces of } k^n\}.$$ 

**Theorem.** As sets, there exists an isomorphism (bijection)

$$kP^n \cong k^n + k^{n-1} + \cdots + k^0.$$ 

These pieces ($k^i$) are called Bruhat cells, but this is sometimes called the Schubert decomposition of $kP^n$, since the closures of the Bruhat cells are called Schubert cells.

**Proof.** Any 1-dimensional subspace of $k^{n+1}$ can be written in the form

$$p = \langle(x_1, x_2, \ldots, x_{n+1})\rangle,$$

where $(x_1, x_2, \ldots, x_{n+1})$ is not the origin. If $x_{n+1} \neq 0$, we can write any such $p$ as

$$p = \left\langle\left(\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, \ldots, \frac{x_{n+1}}{x_{n+1}}\right)\right\rangle = \langle(y_1, y_2, \ldots, y_n, 1)\rangle.$$

There is a clear bijection

$$k^n \cong \{p \in kP^n : p = \langle(y_1, y_2, \ldots, y_n, 1)\rangle\}.$$

Of course, there are those points in $kP^n$ for which $x_{n+1}$ is zero, and are of the form

$$p = \langle(x_1, x_2, \ldots, x_n, 0)\rangle.$$

If $x_n \neq 0$, we can again divide all coordinates by its value to rewrite such $p$ as

$$p = \langle(y_1, y_2, \ldots, y_{n-1}, 1, 0)\rangle.$$

This collection is in bijection with $k^{n-1}$. Via induction, we arrive at our result. \[ \Box \]

2. Some Examples

Since $kP^1 = k + k^0$ is just the affine line plus a single point, we generally write it as

$$kP^1 = k + \{\infty\},$$

which is referred to as “the one point compactification”. Using our coordinate approach, we can write

$$k \cong \{p \in kP^1 : p = \langle(x_1, 1)\rangle\},$$

and

$$\{\infty\} \cong \{p \in kP^1 : p = \langle(x_1, 0)\rangle\}.$$

This infinity is precisely the lines of infinite slope in $k^2$. 

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Example 1. If $k = \mathbb{R}$, then

$$\mathbb{R}P^1 \cong \mathbb{R} + \{\infty\}.$$ 

This one point compactification has the topology of the circle:

Example 2. If $k = \mathbb{C}$, then $\mathbb{C}P^1 \cong \mathbb{C} + \{\infty\} \cong S^2$, again as a topological space. This can be considered as the Riemann sphere (a simply connected compact Riemann surface), as well as a complex variety.
Example 3. Returning to \( k = \mathbb{R} \), we have that

\[
\mathbb{R}P^2 \cong \mathbb{R}^2 + \mathbb{R} + \{\infty\} \\
\cong \mathbb{R}^2 + \mathbb{R}P^1 \\
\cong \mathbb{R}^2 + S^1 \\
\cong S^2 / \sim,
\]
again as a topological space. This is *almost* the disk \( D^2 \). However, \( \mathbb{R}P^2 \) cannot be embedded in \( \mathbb{R}^3 \) without intersection, just like the Klein bottle. Remember that in our original description, we considered points in \( \mathbb{R}P^2 \) via antipodal identification, so we could look at it as the upper half-sphere (which is essentially a disk), with the added requirement that we identify antipodal points at the equator. As a Bruhat decomposition, we have

\[
\begin{tikzpicture}
  \node (infty) at (0,0) {$\{\infty\}$};
  \node (R2) at (2,0) {$\mathbb{R}^2$};
  \node (R) at (4,0) {$\mathbb{R}$};

  \draw[->] (infty) to[bend right] (R2);
  \draw[->] (R2) to[bend right] (R);
  \draw[->] (R) to[bend right] (infty);
\end{tikzpicture}
\]

3. Projective Geometry in Finite Fields

Any finite field has \( q \) elements, where \( q = p^n \) for some prime \( p \). Moreover, all fields with \( q \) elements are isomorphic, so we write \( \mathbb{F}_q \) for “the” field with \( q \) elements. Note that

1. \( \mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z} \) for a prime \( p \);
2. For \( \mathbb{F}_{p^m}, m > 1 \), we take \( \mathbb{F}_p \) and in a sense “throw in” the roots of some irreducible polynomial with coefficients in \( \mathbb{F}_p \).

Question: what is the cardinality of \( \mathbb{F}_q P^n \)? Well, we can use our Bruhat decomposition to find

\[
|kP^n| = |\mathbb{F}_q^n| + |\mathbb{F}_q^{n-1}| + \cdots + |\mathbb{F}_q^0| \\
= q^n + q^{n-1} + \cdots + 1 \\
= q^{n+1} - 1 \quad \frac{q-1}{q-1}.
\]

We call this value, denoted \( [n+1]_q \) as the \( q \)-integer.

Example 4. We call \( \mathbb{F}_2 P^2 \) the Fano plane. By our rule,

\[
|\mathbb{F}_2 P^2| = [3]_2 = 2^2 + 2 + 1 = 7.
\]

If we view \( \mathbb{F}_2^3 \) as \( \langle e_1, e_2, e_3 \rangle \), similar to our natural basis in \( \mathbb{R}^3 \), we can picture the seven points as all possible sums of our basis elements:
If we actually draw them in a plane, we can then consider all possible lines (2-dimensional subspaces) that can be created:

Note that any two lines intersect in precisely one point, while any two points lie on precisely one line.

**Theorem.** In any projective plane $kP^2$:

1. Any two distinct points determine a unique line.
2. Any two distinct lines determine a unique point.

**Proof.** (1) Given any distinct 1-dimensional subspaces $p, p' \in k^3$, the vector space sum

$$p + p' = \{v + v' : v \in p, v' \in p'\}$$

is a 2-dimensional subspace, so it determines a (projective) line. By linear algebra, this line is unique.

(2) Given any distinct 2-dimensional subspaces $\ell, \ell' \in k^3$, we claim $\ell \cap \ell'$ is a 1-dimensional subspace, and therefore a (projective) point. Notice that as vector subspaces,

$$\dim(\ell + \ell') = \dim(k^3) = \dim \ell + \dim \ell' - \dim(\ell \cap \ell')$$

$$\Rightarrow \quad 3 = 2 + 2 - 1.$$  

This (projective) point is again unique by the usual linear algebra.  

Next time we’ll try to axiomatize the concept of projective plane. We’ll use the two properties we just proved, but that won’t quite be enough.