Lie Theory Through Examples
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Lecture 3

1 Representations of Lie Groups

We’ve been having some fun getting lattices from simply-connected complex simple Lie groups. If someone hands us one of these, say $G$, we first choose a maximal compact subgroup $K \subseteq G$. They’re all conjugate inside $G$, so it doesn’t matter which one we choose. Then, we choose a maximal torus $T \subseteq K$. Again, they’re all conjugate inside $K$, so it doesn’t matter which one we choose. Then we can work out the Lie algebra $t$ of $T$ and find a lattice

$$L = \ker e = \subseteq t,$$

where

$$e: t \rightarrow T, \quad x \mapsto \exp(2\pi x).$$

So far we’ve done this for $G = SL(3, \mathbb{C})$ and $K = SL(4, \mathbb{C})$. The other $SL(n, \mathbb{C})$’s work similarly, and soon we’ll do even more examples. It’s a lot of fun.

But, what’s it all good for?

Among other things, it’s good for classifying the complex-analytic representations of $G$, and the unitary representations of $K$. So, we need a word or two about these.

Remember that a representation of any Lie group $H$ is a smooth homomorphism

$$\rho: H \rightarrow GL(V)$$

where $V$ is a vector space and $GL(V)$ is the group of invertible linear transformations of $V$. In what follows we’ll always assume $V$ is finite-dimensional. When $V = \mathbb{C}^n$ we also call $GL(V)$ the general linear group $GL(n, \mathbb{C})$.

Now, the group $GL(V)$ is always a complex manifold: we can cover it with coordinate charts that look like $\mathbb{C}^n$, with complex-analytic transition functions. It makes sense to talk about complex-analytic maps between complex manifolds. And indeed, $GL(V)$ is a complex Lie group: a complex manifold where the functions describing multiplication and inverses are complex analytic. To see this, just use the usual formulas for multiplying and taking inverses of matrices.

If $H$ is a complex Lie group, we say a representation $\rho: H \rightarrow GL(V)$ is complex-analytic if it is complex-analytic as a map between complex manifolds. Such representations are easy to come by:

Exercise 1 Show that $SL(n, \mathbb{C})$ is a complex Lie group, and the obvious representation of $SL(n, \mathbb{C})$ on $\mathbb{C}^n$ is complex analytic.

Exercise 2 Show that if $\rho, \sigma: G \rightarrow GL(V)$ are complex-analytic representations, so are $\rho \oplus \sigma$ and $\rho \otimes \sigma$.

Exercise 3 Show that any subrepresentation of a complex-analytic representation is complex-analytic.

On the other hand, unitary representations are also nice. Given a Lie group $K$ and a finite-dimensional Hilbert space $H$, we define a unitary representation to be a smooth homomorphism:

$$\rho: K \rightarrow U(H)$$

where $U(H)$ is the group of unitary operators on a Hilbert space $H$. When $H = \mathbb{C}^n$ we also call $U(H)$ the unitary group $U(n)$.

Compact Lie groups have lots of unitary representations:
Exercise 4 Suppose $\rho : K \to \text{GL}(V)$ is a (finite-dimensional) representation of a compact Lie group. Show that there is an inner product on $V$ that is invariant under $\rho$, so that letting $H$ denote $V$ made into a Hilbert space with this inner product, we have

$\rho : K \to U(H)$.

The exercises we saw for complex-analytic representations all have analogues for unitary representations:

Exercise 5 Show that $\text{SU}(n)$ is a compact Lie group, and the obvious representation of $\text{SU}(n)$ on $\mathbb{C}^n$ is unitary.

Exercise 6 Show that if $\rho, \sigma : G \to \text{GL}(V)$ are unitary representations, so are $\rho \oplus \sigma$ and $\rho \otimes \sigma$.

Exercise 7 Show that any subrepresentation of a unitary representation is unitary.

But, the really cool part is that when $G$ is a complex simple Lie group and $K$ is its maximal compact subgroup, the complex-analytic representations of $G$ correspond in a one-to-one way to unitary representations of $K$. This fact was called the unitarian trick by Hermann Weyl, who used it to do great things. Let’s state it a bit more precisely:

Theorem 1 Suppose $G$ is a complex simple Lie group and $K$ is its maximal compact subgroup. Given a (finite-dimensional) complex-analytic representation

$\rho : G \to \text{GL}(V)$,

there exists an inner product on $V$ making $\rho|_K$ into a unitary representation. Conversely, given a (finite-dimensional) unitary representation

$\rho : K \to U(H)$,

there exists a unique extension of $\rho$ to a complex-analytic representation of $G$ on the vector space $H$.

2 The Weight Lattice

Now say we have our favorite kind of Lie group $G$: a simply-connected complex simple Lie group. Say someone hands us a complex-analytic representation of $G$. We want to understand it and classify it. By the above theorem, we can think of it as a unitary representation of the maximal compact $K$ without losing any information.

So, let’s do that: say we have unitary representation $\rho$ of $K$ on a finite-dimensional Hilbert space $H$. What do we do now? Since the maximal torus $T$ is a subgroup of $K$, we get a unitary representation of $T$:

$\rho|_T : T \to U(H)$.

And in fact, the maximal torus is big enough that we can completely recover $\rho$ from $\rho|_T$. This is not supposed to be obvious! But it’s great, because unitary representations of tori are incredibly easy to understand.

Here’s how we understand them. Any unitary representation of a torus

$\alpha : T \to U(H)$

can be composed with the exponential map to give a unitary representation of the vector space $t$, thought of as a group:

$\beta = \alpha e.$

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Clearly $\beta$ is trivial on the kernel of $e$:

$$\beta(x) = 1 \text{ for all } x \in L.$$  

Conversely, it’s easy to see a representation

$$\beta: t \to U(H)$$

comes from a representation $\alpha$ as above iff $\beta$ is trivial on $L$. Even better, since the exponential map is onto, knowing $\beta$ tells us $\alpha$. It’s also easy to see that $\beta$ is irreducible iff $\alpha$ is.

So, we just need to understand unitary representations

$$\beta: t \to U(H)$$

that are trivial on $L$. Let’s start with irreducible ones. By Schur’s Lemma, an irreducible representation of an abelian group is 1-dimensional. In this case, $\beta(x)$ is just multiplication by some unit complex number; to be a representation we also need

$$\beta(x + y) = \beta(x)\beta(y).$$

So, it’s easy to see that we get all *irreducible* unitary representations of $t$ from elements $\ell$ of the dual vector space $t^*$, like this:

$$\beta(x) = e^{2\pi i \ell(x)}.$$  

For this to be trivial on $L$ we need

$$\ell(x) \in \mathbb{Z} \text{ for all } x \in L.$$  

This means that $\ell$ needs to lie in the set

$$L^* = \{\ell \in t^*: \ell(x) \in \mathbb{Z} \text{ for all } x \in L\}.$$  

In fact $L^*$ is a lattice in $t^*$ just as $L$ is a lattice in $t$! We call $L^*$ the dual lattice of $L$.

Summarizing what we’ve seen so far:

**Theorem 2** An irreducible unitary representation $\alpha$ of a torus $T$ is specified by choosing a point $\ell$ in the dual lattice $L^*$.

We call the point $\ell \in L^*$ the weight of the representation. When $T$ is the maximal torus of a simply-connected compact simple Lie group $K$, we call $L^*$ the weight lattice of $K$.

But what if $\alpha$ fails to be irreducible? Then it’s a direct sum of irreducible representations — and in an essentially unique way. To specify each of these irreducible representations, we pick a point in $L^*$. We can pick the same point more than once. But we need to pick just finitely many points, since $H$ is finite-dimensional. So:

**Theorem 3** Finite-dimensional unitary representation $\rho$ of a torus $T$ are classified up to unitary equivalence by maps

$$d: L^* \to \mathbb{N}$$

such that

$$\sum_{\ell \in L^*} d(\ell) < \infty.$$  

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The map $d$ says how many times each irreducible unitary representation of $T$ shows up in $\rho$. The theorem says that given any map $d: L^* \to \mathbb{N}$ with $\sum_{\ell \in L^*} d(\ell) < \infty$, there exists a unitary representation $\rho$ with this $d$. Moreover, it says that two unitary representations of $T$ are unitarily equivalent iff they have the same $d$. Here we say $\rho: K \to U(H)$ and $\rho': K \to U(H')$ are \textbf{unitarily equivalent} if there is unitary operator $U: H \to H'$ such that

$$\rho'(k)U = U\rho(k)$$

for all $k \in K$.

For example, say we have a unitary representation of the circle group:

$$\alpha: U(1) \to U(H).$$

Here $u(1) \cong \mathbb{R}$ and $L \subseteq \mathbb{R}$ is just the set of integers. So, to specify a unitary irreducible representation of $U(1)$ we pick an integer $\ell$. Concretely, it goes like this:

$$\alpha(e^{i\theta}) = e^{i\ell \theta}.$$ 

In general, unitary representations of $U(1)$ are classified by maps

$$d: \mathbb{Z} \to \mathbb{N}$$

as in the theorem.