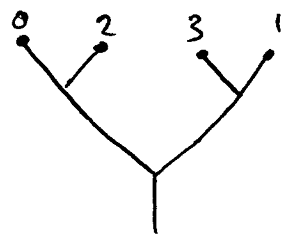
13 Nov 2003

We said a "T-structure" on a finite set S is a way of labeling the leaves of a planar binary tree with elements of S, using each elt. exactly once:

If $S = \{0, 1, 2, 3\}$ here's a T-structure on S:



There are $n!\, c_n$ T-structures on an n-element set, where c_n is the number of n-leaved planar binary trees.

$$|T|(z) = \sum_{n=0}^{\infty} (c_n n!) \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} c_n z^n$$

Now what is c_n?

A T-structure on a set S is: either S being the 1-elt. set XOR writing S as a disjoint union of S' and S'' & putting a T-structure on S' & S''.

$$T = Z + T^2$$

Taking generating functions:

$$|T| = |Z + T^2|$$
$$= |Z| + |T|^2$$
$$= z + |T|^2$$

So: Here's one way to solve this quadratic equation

$$|T| = z + |T|^2$$
$$= z + (z + |T|^2)^2$$
$$= z + (z + (z + |T|^2)^2)^2$$
$$= z + (z + z^2 + 2z|T|^2 + |T|^4)^2$$
$$= z + z^2 + 2z^2 + \dots$$

and we get the beginning of our power series.

Another way to solve a quadratic equation:

$$|T|^2 - |T| + z = 0$$

so

$$|T| = \frac{1 \pm \sqrt{1-4z}}{2}$$

What is this as a power series

$$\sqrt{1+z} = 1 + \tfrac{1}{2} z + \tfrac{1}{2} \cdot \tfrac{-1}{2} \frac{z^2}{2!} + \tfrac{1}{2} \cdot \tfrac{-1}{2} \cdot \tfrac{-3}{2} \frac{z^3}{3!} + \dots$$

$$\sqrt{1-4z} = 1 - 2z - 2^2(1\cdot 1)\frac{z^2}{2!} - 2^3(1\cdot 1\cdot 3)\frac{z^3}{3!} - 2^4(1\cdot 1\cdot 3\cdot 5)\frac{z^4}{4!} - \dots$$

double factorial

Since $c_0 = 0$, $|T|(0) = 0$ so must have $|T|(z) = \frac{1-\sqrt{1-4z}}{2}$:

$$\frac{1-\sqrt{1-4z}}{2} = z + 2\cdot 1\cdot \frac{z^2}{2!} + 2^2(1\cdot 3)\frac{z^3}{3!} + 2^3(1\cdot 3\cdot 5)\frac{z^4}{4!} + 2^4(1\cdot 3\cdot 5\cdot 7)\frac{z^5}{5!} + \cdots$$

$$= z + z^2 + 2z^3 + 5z^4 + 14z^5 + \cdots$$

So we've solved a quadratic equation and counted binary trees:

$$c_n = \frac{2^{n-1}(2n-3)!!}{n!}$$

$$= \frac{(2(n-1))!}{n!(n-1)!}$$

$$= \frac{1}{n}\frac{(2(n-1))!}{(n-1)!(n-1)!}$$

$$= \frac{1}{n}\binom{2(n-1)}{n-1}$$

Note, our c_n is most people's C_{n-1}, i.e.:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

e.g. there are c_3 of these paths , or C_2 of them.

Finally ... What *is* a structure type? It's a certain kind of functor. More precisely it's a functor

the groupoid of finite sets

$$\Phi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$$

Huh? FinSet is the category whose objects are finite sets and whose morphisms are just set functions. Set is the category of *all* set, with functions.

For any category C we get a category C_0 with the same objects as C but only the isomorphisms of C as its morphisms. This is a groupoid, since all morphisms are isomorphisms.

E.g.: FinSet_0 has finite sets as objects and bijections as morphisms

Φ will assign to any finite set S the set $\Phi(S)$ of "Φ-structures on S."

Given two categories C & C', a functor $F: C \to C'$ is a map sending each object $c \in C$ to an object $F(c) \in C'$, and each morphism $f: c_1 \to c_2$ of C to a morphism $F(f): F(c_1) \to F(c_2)$ of C' such that

$$F(fg) = F(f)\,F(g)$$

&

$$F(1_c) = 1_{F(c)}.$$

So, saying $\Phi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$ is a functor says

1) Φ sends any finite set S to a set $\Phi(S)$: the set of Φ-structures on S.

2) Given a bijection $f : S \to S'$ we get a map

$$\Phi(f) : \Phi(S) \longrightarrow \Phi(S')$$

sending Φ-str. on S to Φ-str. on S'.

3) $\Phi(fg) = \Phi(f)\Phi(g)$ says transferring a Φ-str. along fg is same as transferring it along g & then f.

4) $\Phi(1_S) = 1_{\Phi(S)}$ says transferring a Φ-str. along an identity fn. leaves it alone.

What is Z, the structure of "being a 1-elt. set"? It should be a functor

$$Z : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$$

What is it? Given a finite set S, $Z(S) = \begin{cases} \emptyset & \text{if } |S| \neq 1 \\ 1 & \text{if } |S| = 1 \end{cases}$

where 1 is 'the' one-element set.

Given $f: S \to S'$ a bijection

$$Z(f) : Z(S) \to Z(S')$$

is the only possible function; the identity function.

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25 Nov 2003

Structure Types: a <u>structure type</u> is a functor

$$\Phi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$$

where FinSet_0 is the groupoid of finite sets (& bijections) & Set is the category of sets (& functions).

There's a category of structure types: structure types are the objects; what are the morphisms?

$$\begin{tikzcd} \mathrm{FinSet}_0 \arrow[r, bend left=50, "\Phi"{name=U}] \arrow[r, bend right=50, "\Psi"'{name=D}] & \mathrm{Set} \arrow[from=U, to=D, Rightarrow, "T"'] \end{tikzcd}$$

They should be <u>natural</u> <u>transformations</u>.

<u>Def</u>: Given categories C & D, functors $\Phi, \Psi : C \to D$, we define a natural transformation $T : \Phi \Rightarrow \Psi$ to be a function assigning to each object $x \in C$ a morphism $T_x : \Phi(x) \longrightarrow \Psi(x)$ in D s.t. for any morphism $f : x \to y$