$$\widehat{q\psi}(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx}\, x \cdot \psi(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, i \int \frac{d}{dk}(e^{-ikx})\, \psi(x)\, dx$$

by niceness of Schwartz functions

$$= \frac{1}{\sqrt{2\pi}} \cdot i \frac{d}{dk} \int e^{-ikx}\, \psi(x)\, dx$$

$$= \left(i \frac{d}{dk} \hat{\psi}\right)(k)$$

$$= -(p\hat{\psi})(k)$$

So

$$Fq = -pF$$

<u>28 October 2003</u>

We have

$$p, q, F : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

Fourier transform

where

$$\mathcal{S}(\mathbb{R}) = \{\psi : \mathbb{R} \to \mathbb{C} \quad : \quad |x^m \tfrac{d^n}{dx^n} \psi| \text{ bounded}\}$$

& $(p\psi)(x) = -i \frac{d\psi}{dx}(x) \qquad q(\psi)(x) = x\psi(x)$

$$(F\psi)(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx}\, \psi(x)\, dx$$

We saw:

$$Fp = qF$$
$$Fq = -pF$$
$$[p, q] = -i$$

Let's see how this works with the annihilation / creation ops.

Recall
$$a = \frac{q+ip}{\sqrt{2}} \qquad a^* = \frac{q-ip}{\sqrt{2}}$$

Then
$$\begin{aligned} Fa &= \frac{1}{\sqrt{2}} F(q+ip) \\ &= \frac{1}{\sqrt{2}}(Fq + iFp) \\ &= \frac{1}{\sqrt{2}}(-pF + iqF) \\ &= \frac{i}{\sqrt{2}}(qF + ipF) \\ &= \frac{i}{\sqrt{2}}(q+ip)F \\ &= iaF \end{aligned}$$

$$\boxed{Fa = iaF}$$

& similarly:
$$\begin{aligned} Fa^* &= \frac{1}{\sqrt{2}} F(q-ip) \\ &= \frac{1}{\sqrt{2}}(-p-iq)F \\ &= -ia^*F \end{aligned}$$

$$\boxed{Fa^* = -ia^*F}$$

Recall:
$$\psi_0 = e^{-x^2/2}$$
$$\psi_n = a^{*n}\psi_0$$

These have nice simple Fourier transforms; they are eigenvectors

<u>Thm:</u> $F\psi_n = (-i)^n \psi_n$

(In particular, the Gaussian is its own Fourier transform — that's one reason $e^{-x^2/2}$ is the "best" Gaussian)

Proof: We'll show $F\psi_0 = \psi_0$. From this we get

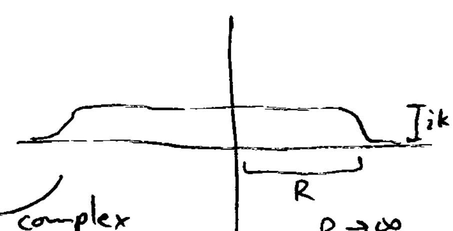
$$
\begin{aligned}
F\psi_n &= F a^{*n} \psi_0 \\
&= (-i)^n a^{*n} F \psi_0 \\
&= (-i)^n a^{*n} \psi_0 \\
&= (-i)^n \psi_n \qquad \text{since } Fa^* = -ia^*F
\end{aligned}
$$

So we only need to check ψ_0.

$$
\begin{aligned}
(F\psi_0)(x) &= \frac{1}{\sqrt{2\pi}} \int e^{-ikx} e^{-x^2/2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x^2 + 2ikx + (ik)^2)} e^{\frac{1}{2}(ik)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \int e^{-\frac{1}{2}(x+ik)^2}\, dx
\end{aligned}
$$

We want to make a change of variables $y = x + ik$. Technically this is illegal because the new variable doesn't run over the real line. We can use a contour integral in $\mathbb{C}$ to show it works rigorously

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \int_{\text{"}-\infty + ik\text{"}}^{\text{"}\infty + ik\text{"}} e^{-\frac{1}{2}y^2}\, dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy}_{\sqrt{2\pi}} \qquad \longleftarrow \text{complex analysis}
\end{aligned}
$$



$$
\begin{aligned}
&= e^{-k^2/2} \\
&= \psi_0(k)
\end{aligned}
$$

Thm: $F: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ extends uniquely to a unitary operator on $L^2(\mathbb{R})$.

Pf: We use the fact that ψ_n form an orthonormal basis of $L^2(\mathbb{R})$, so

$$|n\rangle = \frac{\psi_n}{\|\psi_n\|}$$

is an o.n. basis of $L^2(\mathbb{R})$ with

$$F|n\rangle = (-i)^n |n\rangle$$

BLT Lemma: Suppose V & W are normed vector spaces & W is complete. Suppose $T_0 : V_0 \to W$ is a bounded linear transformation where $V_0 \subseteq V$ is dense. Then T_0 extends uniquely to a bounded lin. op. $T: V \to W$.

Pf: T_0 bounded means

$$\|T_0 \psi\| < K \|\psi\| \qquad \forall \psi \in V_0$$

T is unique since if $\psi \in V$ $\exists \psi_i \to \psi$ for some $\psi_i \in V_0$, so $\|\psi_i - \psi\| \to 0$ so $\|T(\psi_i - \psi)\| < K\|\psi_i - \psi\| \to 0$ so $\|T\psi_i - T\psi\| \to 0$ so $T\psi_i \to T\psi$.

So:

$$T\psi = \lim_i T\psi_i \qquad \text{for any } \psi_i \to \psi \quad \psi_i \in V_0$$

As for existence, we need to check if $\psi_i \to \psi$ & $\psi_i' \to \psi$ then $\lim T\psi_i = \lim T\psi_i'$, so $T\psi$ is well-defined. Also need to show the limit exists:

$\psi_i \to \psi \implies \psi_i$ is Cauchy seq.

$\implies \|T\psi_i - T\psi_j\| \to 0$ since T bdd.

$\Rightarrow T\psi_i$ converges, since W is complete.

So $\lim_{i\to\infty} T\psi_i$ exists

Then we need to check:

① $\lim_{i\to\infty} T\psi_i$ is independent of our choice of $\psi_i \to \psi$.

② $T\psi = \lim_{i\to\infty} T_0\psi_i$ is linear in ψ

③ T is bded.

We won't do this. ∎ back to pf of our thm.

We apply the BLT lemma to $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ — they're dense and extend $F: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ to $F: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ in a unique way s.t. F is bded.

<u>Note</u>: Why did we need to do this by extending from $\mathcal{S}(\mathbb{R})$? Why not just use the formula?

$$\psi(k) = \frac{1}{\sqrt{2\pi}} \int \underbrace{e^{-ikx}\psi(x)}_{\in L^2} dx$$

The reason is that the integral may not exist; we cannot in general integrate an L^2 function. In fact, it turns out that the integral exists for almost all k (w.r.t. Lebesgue measure)

Check that $F: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary. Use the fact that $|n\rangle$ are an o.n. basis to write

$$\psi = \sum_{n=0}^{\infty} a_n |n\rangle \qquad \sum |a_n|^2 < \infty$$

$$\varphi = \sum_{n=0}^{\infty} b_n |n\rangle \qquad \|\psi\|^2$$

To check T is unitary we need to check:

1) $\langle F\psi, F\varphi\rangle = \langle \psi,\varphi\rangle$

2) F must be onto

Why we need this in ∞ dimensions.

$T|n\rangle = |n+1\rangle$

$|n\rangle$ are an o.n. basis of countable dimension v. space

T, the "right shift operator" satisfies 1 but not 2

Note: this is the "Hilbert Hotel Trick"

1) $$\begin{aligned}\langle F\psi, F\varphi\rangle &= \left\langle \sum (-i)^n a_n|n\rangle, \sum(-i)^m b_m|m\rangle \right\rangle \\ &= \sum_n \overline{(-i)^n}\,(-i)^n\,\bar{a}_n b_n \\ &= \langle \psi,\varphi\rangle\end{aligned}$$

2) Given $\psi = \sum a_n|n\rangle \in L^2(\mathbb{R})$, want $\varphi\in L^2(\mathbb{R})$ s.t. $F\varphi = \psi$.

Let
$$\varphi = \sum i^n a_n|n\rangle$$
$$\begin{aligned}F\varphi &= \sum (-i)^n i^n a_n |n\rangle \\ &= \sum a_n|n\rangle = \psi\end{aligned}$$

<u>Cor:</u> $F^{-1}: L^2(\mathbb{R})\to L^2(\mathbb{R})$ has

$$F^{-1}\psi_n = (i)^n\psi_n$$

In fact:
$$(F^{-1}\psi)(x) = \frac{1}{\sqrt{2\pi}}\int e^{ikx}\psi(k)\,dk.$$

To check this in our approach we just need to check that this formula gives this one

Note: in most books, the proof that F^{-1} has this explicit formula is terribly technical and complicated. In our case, the technical part was all compacted into proving that ψ_n form a basis of L^2.

Note: In fact the argument that $F^{-1}\psi$ has this integral formula is the same argument as we used for F, just replacing i by $-i$ (Galois theory!)

$$F\psi_n = (-i)^n \psi_n$$

$$F^2\psi_n = (-1)^n \psi_n \implies (F^2\psi_n)(x) = \psi_n(-x)$$

$$\implies (F^2\psi)(x) = \psi(-x) \quad \forall \psi \in L^2(\mathbb{R})$$

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30 October 2003

The Fourier transform, we saw is a unitary operator:

$$F : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

with

$$F\psi_n = (-i)^n \psi_n$$

where ψ_n are an (unnormalized) basis of eigenvectors of

$$H = \frac{1}{2}(p^2 + q^2)$$

with

$$H\psi_n = (n + \tfrac{1}{2})\psi_n$$

or, these are also eigenvectors of

$$N := H - \frac{1}{2} = \frac{1}{2}(p^2+q^2-1) = a^*a$$

(This amounts to picking a new origin for energy)

with

$$N\psi_n = n\psi_n$$

Time Evolution:

According to Heisenberg, observables evolve in time: for any observable (s.a. operator on Hilbert space) O, and any time $t \in \mathbb{R}$, there is an observable $O(t)$ which is "measuring O after waiting time t."
$O(t)$ satisfies Heisenberg's equation:

$$\frac{dO(t)}{dt} = i[H, O(t)]$$

where H is the <u>energy</u> or Hamiltonian. The solution is

$$O(t) = e^{itH} O \, e^{-itH}$$

There are also states ψ (unit vectors in the Hilbert space) — but Heisenberg didn't describe how these change in time. The Heisenberg picture of a <u>state</u> is that it describes the system for all time — that way we only need to worry about $O(t)$, not "$\psi(t)$." The expectation value of $O(t)$ in state ψ is

$$\langle \psi, O(t)\psi \rangle$$

According to Schrödinger, states (that is, the vectors) evolve in time, while observables stay fixed. States evolve according to Schrödinger's equation: Given a state ψ at time zero, we get a state $\psi(t)$ at time t satisfying

$$\frac{d\psi(t)}{dt} = -iH\psi(t)$$

The solution is:

$$\psi(t) = e^{-iHt}\psi$$

Schrödinger says the expectation value of an observable O in state $\psi(t)$ is

$$\langle \psi(t), O\,\psi(t) \rangle$$

In fact, Heiseberg & Schröding pictures agree:

$$\begin{aligned} \langle \psi(t), O\psi(t)\rangle &= \langle e^{-itH}\psi, Oe^{-itH}\psi\rangle \\ &= \langle \psi, (e^{-itH})^{*} O\, e^{-itH}\psi\rangle \\ &= \langle \psi, e^{itH} O e^{-itH}\psi\rangle \\ &= \langle \psi, O(t)\psi\rangle . \end{aligned}$$

Now: what's e^{-itH} like for the harmonic oscillator?

Or: what's e^{-itN} like?

$$e^{-itN}\psi_n = \sum_{k=0}^{\infty} \frac{(-itN)^k}{k!}\psi_n$$

and using $N\psi_n = n\psi_n$,

$$e^{-itN}\psi_n = \sum_{k=0}^{\infty} \frac{(-itn)^k}{k!}\psi_n$$

$$= e^{-itn}\psi_n .$$

Recall:

$$F\psi_n = (-i)^n \psi_n$$

These are the same when $e^{-it} = -i$, e.g. $t = \frac{\pi}{2}$!

(Upshot: "we can take Fourier transforms just by waiting around.")

So

$$F = e^{-i\frac{\pi}{2}N}$$

In short:

"To take the Fourier transform of a function, get a particle whose ψ is that function & wait a quarter period of the harmonic oscillator $(\frac{1}{4}2\pi)$" (with ground state energy subtracted off)

This is the time for "momentum energy" to turn into "position energy" (kinetic into potential)

Or:

$$F = e^{-i\frac{\pi}{2}N} = (-i)^N .$$

Note: $F^2 \psi_n = (-1)^n \psi_n$

ψ_n with n even form a basis of <u>even</u> L^2 functions
ψ_n with n odd form a basis of <u>odd</u> L^2 functions

So: for any $\psi \in L^2(\mathbb{R})$

$$(F^2\psi)(x) = \psi(-x)$$
$$= (P\psi)(x)$$

for the <u>parity</u> operator P

So "The Fourier Transform is a square root of Parity."

Next, note:

$$F^4 \psi_n = \psi_n$$

so

$$F^4 = \mathbb{1}$$

What's going on? Why is the Fourier transform like a quarter turn?

Answer:

$$\begin{aligned} q(t) &= \cos t \; q + \sin t \; p \\ p(t) &= -\sin t \; q + \cos t \; p \end{aligned} \qquad (*)$$

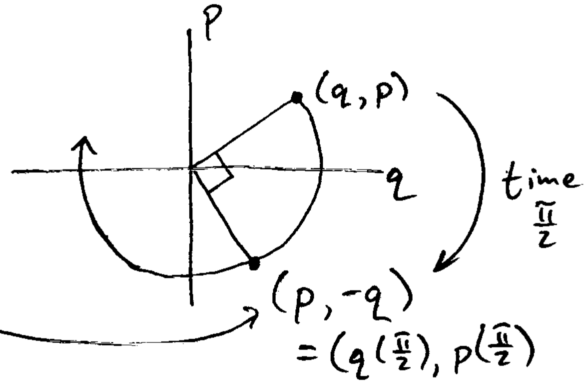
We saw this before with

$$q(t) = e^{+itH} q e^{-itH} = e^{it(H-\frac{1}{2})} q e^{-it(H-\frac{1}{2})} = e^{itN} q e^{-itN}$$

and similarly

$$p(t) = e^{itH} p e^{-itH} = e^{itN} p e^{-itN}$$

Think of this in the classical picture



got this by using (*) on previous page

time evolution by $\frac{\pi}{2}$ is a quarter turn in phase space.

In the quantum picture: we can plug $\frac{\pi}{2}$ in to our eqns. for $q(t)$, $p(t)$ above to get

$$p = q\left(\tfrac{\pi}{2}\right) = e^{i\frac{\pi}{2}N} q e^{-i\frac{\pi}{2}N} = F^{-1} q F \implies \boxed{Fp = qF}$$

$$-q = p\left(\tfrac{\pi}{2}\right) = e^{i\frac{\pi}{2}N} q e^{-i\frac{\pi}{2}N} = F^{-1} p F \implies \boxed{Fq = -pF}$$

as before

$\delta_a(x)$ – position is perfectly known:

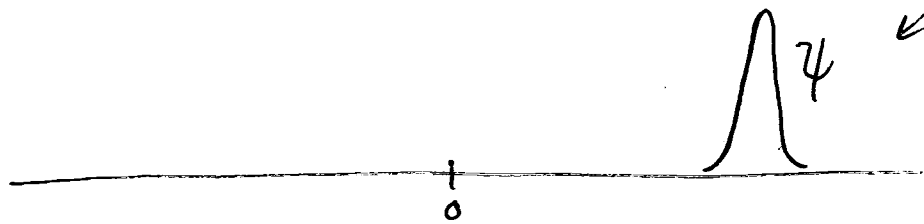
$$q\,\delta_a(x) = a\,\delta_a(x)$$

$$F\delta_a(x) = \frac{e^{-iax}}{\sqrt{2\pi}}$$

$$p\,\frac{e^{-iax}}{\sqrt{2\pi}} = a\,\frac{e^{-iax}}{\sqrt{2\pi}}$$

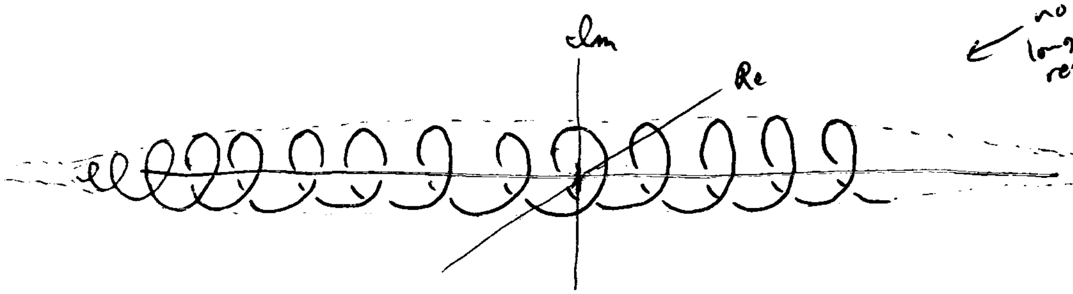
– momentum is perfectly well known

The quantum harmonic oscillator



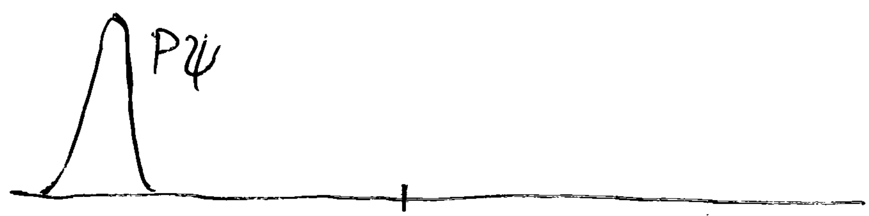
← real valued

$t = 0$



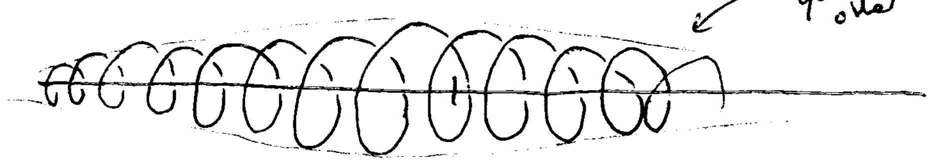
← no longer real valued

$t = \frac{\pi}{2}$

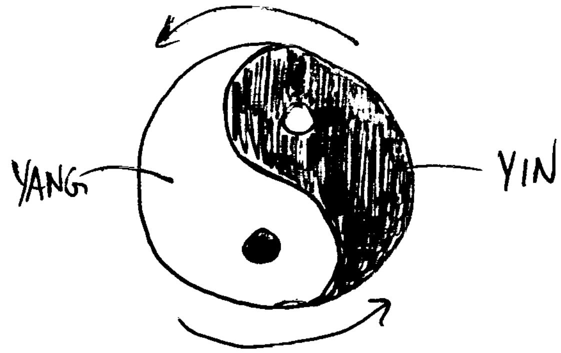


$t = \pi$

corkscrew goes the other way

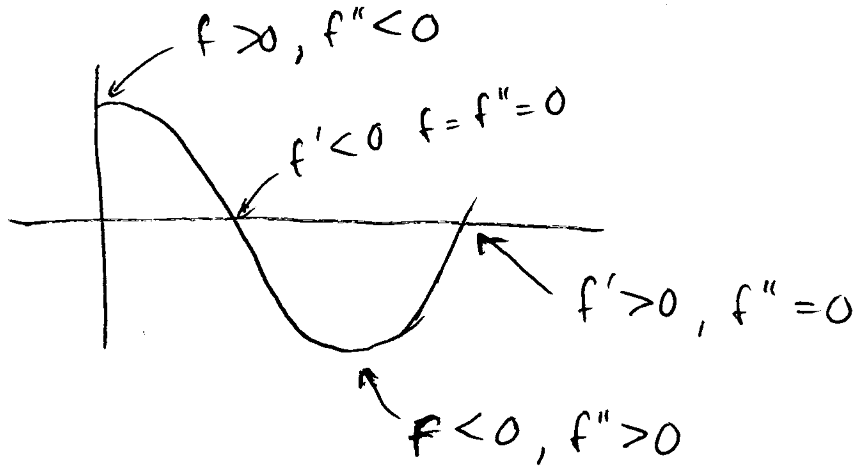


The Mathematics of Tao

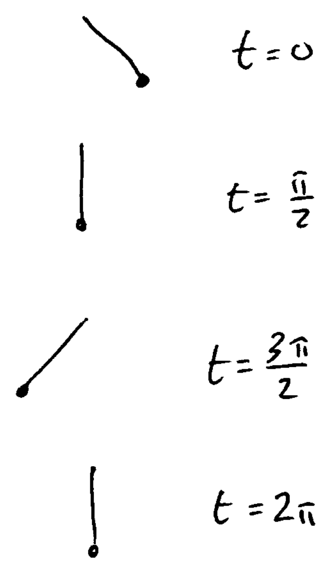


YANG = + = HOT = SUNNY

YIN = − = COLD = DARK



4 most interesting points in a cyclic process



4 November 2003

Let's summarize what we've done so far:

We have a space with basis $\psi_0, \psi_1, \psi_2, \ldots$ on which various operators act:

creation: $a^* \psi_n = \psi_{n+1}$

annihilation: $a \psi_n = n \psi_{n-1}$

position: $q = \frac{a + a^*}{\sqrt{2}}$

momentum: $p = \frac{a - a^*}{\sqrt{2}\, i}$

Hamiltonian: $H = \frac{1}{2}(p^2 + q^2)$ $\qquad H\psi_n = (n + \frac{1}{2})\psi_n$

renormalized Hamiltonian (subtracting off ground state energy): or "number operator" $N = H - \frac{1}{2} = a^* a$ $\qquad N\psi_n = n\psi_n$

Fourier Transform: $F = (-i)^N = e^{-itN}$ where $t = \frac{\pi}{2}$ - a quarter period. $\qquad F\psi_n = (-i)^n \psi_n$

In terms of the Fock representation we call this vector space cx. polynomials in one variable, i.e. $\mathbb{C}[z]$. We call ψ_n "z^n". Thus we get:

$$a^* = M_z \qquad \text{(multiplication by } z\text{)}$$

$$a = \frac{d}{dz}$$

$$N = M_z \frac{d}{dz} \qquad (N z^n = z\, n z^{n-1} = n z^n)$$