Thm: If we normalize ψ_n to get

$$|n\rangle = \frac{\psi_n}{\|\psi_n\|}$$

these form an o.n. basis of $L^2(\mathbb{R})$.

Pf: They're orthonormal, since the ψ_n are orthogonal. They form a basis since the finite lin. combs. of the ψ_n are dense in $\mathcal{S}(\mathbb{R})$ & thus by previous prop. in $L^2(\mathbb{R})$. ∎

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21 October 2003

The Weyl Algebra: the assoc. alg. gen by p, q s.t.

$$[p,q] = -i.$$

is also generated by

$$a = \frac{q+ip}{\sqrt{2}}$$

$$a^* = \frac{q-ip}{\sqrt{2}}$$

with $[a,a^*] = 1$, since

$$\frac{a+a^*}{\sqrt{2}} = q$$

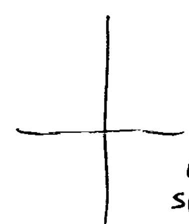
$$\frac{a-a^*}{\sqrt{2}\,i} = p$$

note the similarity with

$z = x+iy$
$\bar{z} = x-iy$

$\frac{z+\bar{z}}{2} = x$
& $\frac{z-\bar{z}}{2i} = y$

$\mathbb{R}^2 \cong \mathbb{C}$



is phase space for classical harm. oscillator

We considered the Schrödinger rep of the Weyl algebra, where

$$p, q : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

$$\mathcal{S}(\mathbb{R}) = \left\{ \psi : \mathbb{R} \to \mathbb{C} \;:\; \left| x^n \frac{d^n}{dx^n} \psi \right| \text{ bded} \right\}$$

Now we're doing a quantum version of the same thing: a, a^* are like z & $\bar{z}$

($\to$ noncommutative complex analysis)

and

$$p = -i\frac{d}{dx}$$

$$q = M_x \qquad \text{(mult. by } x\text{)}$$

We considered the eigenfunctions of

$$H = \frac{1}{2}(p^2+q^2) : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

& saw they were

$$\psi_0 = e^{-x^2/2} \qquad H\psi_0 = \frac{1}{2}\psi_0$$

and

$$\psi_n = a^{*n}\psi_0 \qquad H\psi_n = (n+\tfrac{1}{2})\psi_n .$$

In fact if we normalize these:

$$|n\rangle = \frac{\psi_n}{\|\psi_n\|} \qquad \|\psi\|^2 = \int_{\mathbb{R}} |\psi|^2\,dx$$

we get an o.n. basis of $L^2(\mathbb{R})$. We saw

$$a^*\psi_n = \psi_{n+1}$$

$$a\,\psi_n = n\,\psi_{n-1}$$

$$H\psi_n = (n+\tfrac{1}{2})\psi_n$$

As operators on $\mathcal{S}(\mathbb{R})$ we thus have

$$H = a^*a + \tfrac{1}{2}I.$$

In the Weyl algebra we indeed have $H = a^*a + \frac{1}{2}$:

$$a^*a = \left(\frac{q-ip}{\sqrt{2}}\right)\left(\frac{q+ip}{\sqrt{2}}\right)$$

$$= \frac{1}{2}(q-ip)(q+ip)$$

$$= \frac{1}{2}(q^2 \underbrace{- ipq + iqp}_{-i[p,q]=-1} + p^2) = \frac{1}{2}(p^2+q^2-1)$$

$$= H - \frac{1}{2} \quad \text{the harmonic oscillator minus } \tfrac{1}{2}.$$

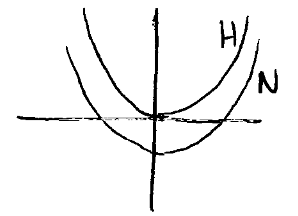
It would be nice to have everything in our eqns. be natural numbers. Why? Because we know how to categorify $\mathbb{N}$ as FinSet. We have:

$$H\psi_n = (n+\tfrac{1}{2})\psi_n$$

the fly in the ointment – an unnatural number.

To categorify we modify the Hamiltonian:

$$N = H - \tfrac{1}{2}$$
$$= a^*a.$$



If we let N, the number operator be the "new, improved" Hamiltonian for our oscillator, we're really replacing

$$\tfrac{1}{2}(p^2+\underbrace{q^2}_{\text{potential}}) \quad \text{by} \quad \tfrac{1}{2}(p^2+\underbrace{q^2-1}_{\text{potential}})$$

This doesn't change any physics since:

1) classically: $F = -V'$ doesn't change if you add a constant to V

2) quantumly $\frac{dO}{dt} = i[H,O]$ doesn't change if you add a constant to H.

So we get a rep. of the Weyl algebra with

$$\begin{cases} a^*\psi_n = \psi_{n+1} \\ a\psi_n = n\,\psi_{n-1} \end{cases}$$

(and $N\psi_n = n\psi_n$) $\quad (N = a^*a)$

We call this the Heisenberg representation of the Weyl algebra if we treat ψ_n as abstract basis vectors, not wavefunctions.

BUT we can also think of the Heisenberg rep this way

$$\psi_n = z^n$$
$$a = \frac{d}{dz}$$
$$a^* = M_z$$

In short, we get a rep on the Weyl algebra on $\mathbb{C}[z]$ by these formulas. This is called the Fock Representation. If we think of $a = \frac{d}{dz}$ & $a^* = M_z$ as operators on polynomials, or analytic functions, or ..., on the complex plane, we call this the Bargmann - Segal representation.

We will categorify the Fock representation & see this is related to combinatorics. But before we get into that ...

We want to talk about:
(1) The Uncertainty Principle
(2) Fourier Transform.

(1). Suppose A is a self-adjoint operator on a Hilbert space H, finite-dim. for simplicity. We think of A as an "observable", but how does this work? We think of unit vectors $\psi \in H$ as states: ways our system can be. We say $|\langle\psi,\varphi\rangle|^2$ is the probability of finding system in state ψ given that it is in state φ. Note this number doesn't change if we multiply ψ by a phase ($c \in \mathbb{C}$ s.t. $|c| = 1$), so really states are equivalence classes of unit vectors mod phase.

If A is s.a. then H has an o.n. basis of eigenvectors $e_i \in H$

$$Ae_i = \lambda_i e_i$$

The numbers λ_i are all of the possible outcomes of measuring A, & ~~the~~ e_i is the state in which measuring A always yields the value λ_i. Suppose we measure A in some arbitrary state ψ. We can write

$$\psi = \sum \langle e_i, \psi \rangle e_i$$

where $|\langle e_i, \psi \rangle|$ is prob. of system in state ψ to be found in state e_i. So when we measure A in state ψ we get the answer λ_i with prob $|\langle e_i, \psi \rangle|^2$. These probabilities sum to 1:

$$\sum |\langle e_i, \psi \rangle|^2 = \|\psi\|^2 = 1$$

The <u>mean</u> or <u>expected</u> value of A in state ψ:

conj. linear, linear

$$\begin{aligned}
\sum |\langle e_i, \psi \rangle|^2 \lambda_i &= \sum \langle \psi, e_i \rangle \langle e_i, \psi \rangle \lambda_i \\
&= \sum \langle \psi, Ae_i \rangle \langle e_i, \psi \rangle \\
&= \sum \langle A\psi, e_i \rangle \langle e_i, \psi \rangle \\
&= \langle A\psi, \psi \rangle \\
&= \langle \psi, A\psi \rangle
\end{aligned}$$

23 October 2003

A quantum system has some Hilbert space H s.t.

1) **states** are unit vectors $\psi \in H$ (really equivalence classes where $\psi \sim \psi'$ if $\psi' = c\psi$ for some $c \in \mathbb{C}$ with $|c| = 1$)

2) **observables** are self-adjoint operators A on H.

If A has an o.n. basis of eigenvectors

$$Ae_i = \lambda e_i$$

then

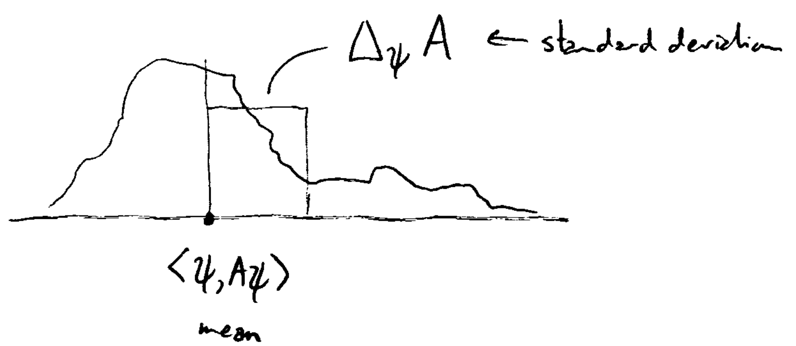
i) when you measure A in state e_i you always get λ_i.

ii) when you measure A in some state ψ you get λ_i with prob. $|\langle e_i, \psi\rangle|^2$.

Thus the *mean value* (*expected value* or *expectation value*) of A in state ψ

$$\sum_i |\langle e_i, \psi\rangle|^2 \lambda_i = \langle \psi, A\psi\rangle$$

(Note: when we measure A, the state is not then in state "$A\psi$". A doesn't take a state to a new state – that's what *unitary* operators do in QM.)



The standard deviation of some quantity x is the ~~mean of~~ square root of the mean of

$$(x - \text{mean of } x)^2$$

note: can't use just mean of $(x-\bar{x})$ that's zero!

(The RMS)

The mean of $(x - \text{mean of } x)^2$ is the <u>variance</u> $(\Delta_\psi \cancel{X})^2$; standard deviation is Δx.

So in quantum mechanics the variance of A in state ψ is

$$(\Delta_\psi A)^2 = \langle \psi, (A - \langle \psi, A\psi \rangle)^2 \psi \rangle$$

1) <u>Thm:</u> (Uncertainty Principle)

Suppose A, B are self-adjoint operators on a Hilbert space H, and $\psi \in H$ is a unit vector. Then

$$\Delta_\psi A \, \Delta_\psi B \geq \frac{1}{2} |\langle \psi, [A,B]\psi \rangle|$$

<u>Proof:</u> First we'll modify A & B a little. Let

$$A' = A - \langle \psi, A\psi \rangle I$$
$$B' = B - \langle \psi, B\psi \rangle I$$

(i.e. subtract of the means)

so they have mean zero:

$$\langle \psi, A'\psi \rangle = \langle \psi, (A - \langle \psi, A\psi \rangle)\psi \rangle = \langle \psi, A\psi \rangle - \langle \psi, A\psi \rangle \langle \psi, \psi \rangle = 0$$

& $\langle \psi, B'\psi \rangle = 0$

Note $[A', B'] = [A,B]$ and also the standard deviations are unaffected:

$$\Delta_\psi A' = \Delta_\psi A$$
$$\Delta_\psi B' = \Delta_\psi B$$

since for any constant C

$$\Delta_\psi (A+C) = \langle \psi, (A+C-\langle \psi, (A+C)\psi\rangle)^2 \psi\rangle$$
$$= \langle \psi, (A - \langle \psi, A\psi\rangle)^2 \psi\rangle$$
$$= \Delta_\psi A.$$

Note: since mean of A' is zero,

$$(\Delta_\psi A')^2 = \langle \psi, A'^2 \psi\rangle$$

and same for B. It suffices to show the uncertainty principle for A' & B', i.e. we need to show $\sqrt{\langle \psi, A'^2\psi\rangle}\sqrt{\langle \psi, B'^2\psi\rangle} \geq \frac{1}{2}|\langle \psi, [A', B']\psi\rangle|$

$$\langle \psi, A'^2 \psi\rangle \langle \psi, B'^2\psi\rangle \geq \frac{1}{4}|\langle \psi, [A', B']\psi\rangle|^2$$

$$\|A'\psi\|^2 \cdot \|B'\psi\| \geq \frac{1}{4}|\langle \psi, A'B'\psi\rangle - \langle \psi, B'A'\psi\rangle|^2$$

$$\frac{1}{4}|\langle A'\psi, B'\psi\rangle - \langle B'\psi, A'\psi\rangle|^2$$

So it suffices to show

$$\frac{1}{4}|\langle A'\psi, B'\psi\rangle - \langle B'\psi, A'\psi\rangle|^2 \leq \frac{1}{4}|2\langle A'\psi, B'\psi\rangle|^2$$
$$= |\langle A'\psi, B'\psi\rangle|^2$$
$$\leq \|A'\psi\|^2 \|B'\psi\|^2 \quad \blacksquare$$

complex conjugates

Note: Cauchy Schwarz $\Leftrightarrow$ Uncertainty Principle.

Note: this is all fine if A, B bded or if ψ is in the domain of A', B', $A'B'$ and $B'A'$.

Cor: If $\psi \in \mathcal{S}(\mathbb{R})$ then

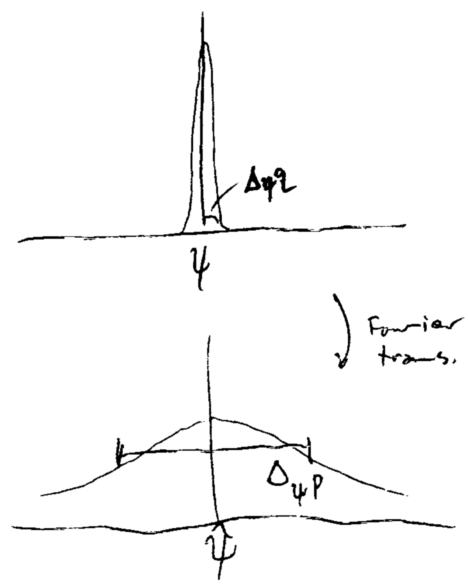
$$\Delta_\psi p \, \Delta_\psi q \geq \frac{1}{2}$$

Pf: in the previous thm, the R.H.S. was

$$\frac{1}{2} \left| \langle \psi, \underbrace{[p,q]}_{=-i} \psi \rangle \right|^2 = \frac{1}{2} \quad \blacksquare$$

Or, restoring $\hbar$:

$$\boxed{\Delta_\psi p \, \Delta_\psi q \geq \frac{\hbar}{2}}$$



1) This inequality is sharp! (can't make it better and have it still be true.) (i.e. we can get equality)

2) There are lots of ψ that get equality. You can write all of them down. The simplest is the g.s. of the harmonic oscillator: $\psi = \dfrac{e^{-x^2/2}}{\| e^{-x^2/2} \|_{L^2}}$

<u>Thm:</u> If ψ is an eigenvector of the harmonic oscillator hamiltonian

$$H = \frac{1}{2}(p^2+q^2)$$

$$H\psi = E\psi$$

then $$E \geq \frac{1}{2}.$$

(Note: for $\psi = \frac{e^{-x^2/2}}{\|e^{-x^2/2}\|}$ we do have $H\psi = \frac{1}{2}\psi$)

<u>Proof:</u> By the <u>proof</u> of the uncertainty principle,

$$\langle \psi, p^2\psi\rangle\langle\psi, q^2\psi\rangle \geq \underbrace{\frac{1}{4}|\langle\psi, \overbrace{[p,q]}^{-i}\psi\rangle|^2}_{=\frac{1}{4}}$$

(really just Cauchy-Schwarz for p, q s.a.)

But if $H\psi = E\psi$,

$$\langle\psi, \tfrac{1}{2}(p^2+q^2)\psi\rangle = E$$

so we need

$$\langle\psi, p^2\psi\rangle + \langle\psi, q^2\psi\rangle \geq 1.$$

<u>Lemma:</u> If $A, B \geq 0$ and $AB \geq \frac{1}{4}$ then $A+B \geq 1$ if $A, B \geq 0$.

<u>Pf:</u> $AB \geq \frac{1}{4}$ so $B \geq \frac{1}{4A}$ so $A+B \geq A + \frac{1}{4A}$

Why is $A + \frac{1}{4A} \geq 1$? Find min $A = \frac{1}{2}$ & check $A + \frac{1}{4A} = 1$ at minimum. ∎

2) Fourier Transform

The Fourier transform:

$$F : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

is given by:

$$F\psi = \hat{\psi}$$

where

$$\hat{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \psi(x) \, dx$$

This relates the momentum and position operators nicely:

Example:

$$\begin{aligned}
\widehat{(p\psi)}(k) &= \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \cdot -i\psi'(x) \, dx \\
&= \frac{1}{\sqrt{2\pi}} \int i(-ike^{-ikx})\psi(x) \, dx \\
&= \frac{1}{\sqrt{2\pi}} k \int e^{-ikx} \psi(x) \, dx \\
&= k\,\hat{\psi}(k) \\
&= (q\hat{\psi})(k)
\end{aligned}$$

integration by parts. No boundary terms because Schwartz fns. are so nice

So

$$\boxed{Fp = qF}$$

$$\widehat{q\psi}(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx}\, x \cdot \psi(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\, i \int \frac{d}{dk}(e^{-ikx})\, \psi(x)\, dx$$

by niceness of Schwartz functions

$$= \frac{1}{\sqrt{2\pi}} \cdot i \frac{d}{dk} \int e^{-ikx}\, \psi(x)\, dx$$

$$= \left(i \frac{d}{dk} \hat{\psi}\right)(k)$$

$$= -(p\hat{\psi})(k)$$

So

$$Fq = -pF$$

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28 October 2003

We have

$$p, q, F : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

Fourier transform

where

$$\mathcal{S}(\mathbb{R}) = \{\psi : \mathbb{R} \to \mathbb{C} \quad : \quad |x^m \tfrac{d^n}{dx^n} \psi| \text{ bounded}\}$$

& $\quad (p\psi)(x) = -i \frac{d\psi}{dx}(x) \qquad q(\psi)(x) = x\,\psi(x)$

$$(F\psi)(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx}\, \psi(x)\, dx$$

We saw:

$$Fp = qF$$
$$Fq = -pF$$
$$[p, q] = -i$$