$$[p,q^2] = [p,q]q + q[p,q] = -2iq$$

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14 October 2003

JB forgot his notebook today...

The Weyl algebra is the associative alg. over $\mathbb{C}$ generated by p, q satisfying

$$pq - qp = -i$$

We'll do calculations in this algebra (or some larger algebra containing things like e^{itH} where H is an elt. of Weyl algebra). Starting with:

$$[p,q] = -i$$
$$[p,q^2] = [p,q]q + q[p,q] = -2iq$$
$$\vdots$$
$$[p,q^n] = -inq^{n-1}$$

or for any polynomial F,

$$[p, F(q)] = -i F'(q)$$

or for short

$$[p, \cdot\,] = -i\frac{\partial}{\partial q}.$$

Next:

$$[q,p^2] = [q,p]p + p[q,p] = 2ip$$
$$[q,p^n] = inp^{n-1}$$

or for any poly. F

$$[q, F(p)] = iF'(p)$$

or:

$$[q, \cdot\,] = i\frac{\partial}{\partial p}.$$

If $H = \frac{1}{2}p^2 + V(q)$ (V a poly. or perhaps an analytic fn. (power-series))

then we have

$$\frac{d}{dt}O(t) = [iH, O(t)]$$

where $O(t) = e^{itH} O e^{-itH}$

We are doing QM in a nonstandard more algebraic way to show that all of elementary QM is really the result of $[p,q] = -i$.

In particular, for the observable q,

$$\frac{d}{dt}q(t) = \frac{d}{dt}\left(e^{itH} q e^{-itH}\right)$$
$$= e^{itH}(iHq - qiH)e^{-itH}$$
$$= e^{itH}[iH, q]e^{-itH}$$

← note: this is general. e.g. $\frac{d}{dt}p(t) = e^{itH}[iH,p]e^{-itH}$.

Now $[iH, q] = i\left[\frac{p^2}{2} + V(q), q\right]$

$[V(q), q] = 0$ since V is a poly. in q and the alg. is associative

$$= \frac{i}{2}[p^2, q]$$
$$= \frac{i}{2} \cdot -2ip$$
$$= p$$

So $$\frac{d}{dt}q(t) = e^{itH} p e^{-itH} = p(t)$$

i.e. $$\dot{q}(t) = p(t)$$ ← "velocity = momentum" (when mass = 1)

Next, try

$$[iH, p] = i\left[\frac{p^2}{2} + V(q), p\right]$$
$$= i[V(q), p]$$
$$= i \cdot i V'(q)$$
$$= -V'(q)$$ ← force at time 0.

so

$$\frac{d}{dt}p(t) = e^{itH}(-V'(q))e^{-itH} = -V'(e^{itH} q e^{-itH}) = -V'(q(t))$$

↖ if V is a poly, do this term by term.

In short

$$\dot{p}(t) = -V'(q(t)) \qquad \leftarrow \text{“time derivative of momentum is force”}$$

These give

$$\ddot{q}(t) = -V'(q) \qquad \text{“}F = a\text{”}$$

or if we put m back in & define $F(t) = -V'(q(t))$ – the force operator – ~~accel~~ and ~~the~~ $a(t) = \ddot{q}(t)$ – the acceleration operator, we get

$$F(t) = m a(t)$$

(For some reason, nobody talks about where F=ma goes in QM. JB has spent lots of time on spr. trying to convince people this is how it works.)

For the harmonic oscillator:

$$H = \tfrac{1}{2}(p^2 + q^2)$$

& then

$$\dot{q}(t) = p(t) \qquad \dot{p}(t) = -q(t)$$

$$\& \quad \ddot{q}(t) = -q(t)$$

note: these look just like the eqs we wrote down for the classical case. They are still valid in QM.

These have solutions

$$\left.\begin{aligned} q(t) &= (\cos t)q + (\sin t)p \\ p(t) &= -(\sin t)q + (\cos t)p \end{aligned}\right]$$

same formula as classically! (but it means something different)

(JB is trying to give the impression that QM is "just like" CM except done in a Weyl algebra instead of a Poisson algebra.)

To go further, we want to look at representations of the Weyl algebra as linear operators, i.e. some vector space V & linear operators $p, q : V \to V$ s.t. $pq - qp = \underbrace{-i}_{-i 1_V}$.

You could hope for a finite-dim V, e.g. $V = \mathbb{C}^n$ so then $p, q \in M_n(\mathbb{C})$. (Matrix mechanics!) In fact there are <u>no</u> fin. dim. reps except for the zero-dim rep $n=0$.

<u>Proof:</u> Suppose $p, q \in M_n(\mathbb{C})$, then

$$0 = tr(pq - qp) = tr(-i\mathbb{1}) = -i \dim(V) = -ni$$

So we conclude $n=0$ (toby's rk. if we kept $\hbar$ in, then the other alternative would be $\hbar = 0$). ∎

Next: try to find some normed vector space V s.t. $p, q : V \to V$ are bounded lin. ops. on V. In fact, there are no reps like this except the zero dimensional one.

<u>Proof:</u>

$$\begin{aligned} n\|q^{n-1}\| = \|-inq^{n-1}\| = \|[p, q^n]\| &= \|pq^n - q^n p\| \\ &\leq \|pq^n\| + \|q^n p\| \\ &\leq \|pq\| \cdot \|q^{n-1}\| + \|qp\| \cdot \|q^{n-1}\| \end{aligned}$$

So

$$n \leq \|pq\| + \|qp\| \quad \forall n$$

— a contradiction (since p & q are bded) unless $\|q^{n-1}\| = 0$, in which case $\|q^n\| \leq \|q\| \cdot \|q^{n-1}\| = 0$ so $q^n = 0$ so $\|[p, q^n]\| = 0$ so $\|q^{n-1}\| = 0$ so $q^{n-1} = 0$ ... so $q = 0$. ∎

So $1_V = 0$ so $\dim V = 0$.

So: we can try to describe p & q as <u>unbounded operators</u> on a Hilbert space. The classic example is called the <u>Schrödinger representation</u> of the Weyl algebra.

Idea: to let p&q be operators on some space of fns $\psi : \mathbb{R} \to \mathbb{C}$.

$$(p\psi)(x) = -i\psi'(x) \qquad \text{or} \quad p = -i\frac{\partial}{\partial x}$$

$$(q\psi)(x) = x\psi(x) \qquad\qquad q = M_x$$

$\uparrow$ mult. by x

$$p(q\psi)(x) - q(p\psi)(x) = -i\frac{d}{dx}(x\psi)(x) + i\,x\frac{d\psi}{dx}(x) =$$

$$-i\psi(x) - ix\frac{d\psi}{dx}(x) + ix\frac{d\psi}{dx}(x) = -i\psi(x)$$

The idea: the fn ψ is a "wavefunction" that describes the probability of finding the particle in some set $S \subseteq \mathbb{R}$: the probability is

$$\int_S |\psi(x)|^2\, dx$$

(if ψ is normalized so that $\int_{\mathbb{R}} |\psi(x)|^2 = 1$)

(The Plan:)

To write p&q as $\infty \times \infty$ matrices, as Heisenberg did, we'll take Schrödinger rep & pick a basis of functions on the real line — a basis of eigenvectors of

$$H = \frac{1}{2}(p^2 + q^2)$$

$$= \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$$

$\uparrow$ the operator "mult. by x^2"

What kind of fns on the real line? One choice: Schwartz functions $\mathcal{S}(\mathbb{R}) = \{\psi : \mathbb{R} \to \mathbb{C} : \; \left|x^m \frac{d^n}{dx^n}\psi(x)\right| < C\}$

$\forall n, m,\ \exists C$ s.t.

"All derivatives of ψ exist & vanish faster than the reciprocal of any polynomial."

<u>Note</u>: Schwartz functions satisfy both of the nice properties in analysis:

- smoothness
- fast decay

] note: these are <u>dual</u> notions. The Fourier transform takes smooth fns. to ones that decay fast & vise versa.

$$\mathcal{S}(\mathbb{R}) = \{\psi : \quad q^m p^n \psi \text{ is bounded } \forall n,m\}$$
$$= \{\psi : \quad p^n q^m \psi \text{ is bounded } \forall n,m\}.$$

We will find linearly independent eigenfns of H, say ψ_i, s.t. finite lin. combs are dense in the natural topology on $\mathcal{S}(\mathbb{R})$.

(These are in practice Hermite polynomials times gaussians.)

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16 October 2003

Let, the Schwartz space, $\mathcal{S}(\mathbb{R}) = \{\psi : \mathbb{R} \to \mathbb{C} : x^m \frac{d^n\psi}{dx^n} \text{ is bounded } \forall n,m\}$

Given a sequence or net $\psi_\alpha \in \mathcal{S}(\mathbb{R})$ we say $\psi_\alpha \to \psi \in \mathcal{S}(\mathbb{R})$ if

$$\sup \left| x^m \frac{d^n}{dx^n}(\psi_\alpha - \psi)\right| \longrightarrow 0 \quad \forall n,m$$

making $\mathcal{S}(\mathbb{R})$ into a topological vector space.

We have cts. lin. ops

$$p, q : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

given by

$$(p\psi)(x) = -i\psi'(x)$$
$$(q\psi)(x) = x\psi(x)$$

(Easy Exercise: check that these are continuous)

& also

$$H : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

given by

$$H = \frac{1}{2}(p^2 + q^2)$$

note: p & q have no eigenvectors in this space! They should look like δ-fns. and these clearly aren't in $\mathcal{S}$.

Q: What are the eigenvectors of H?

(Note: p & q don't have eigenvectors in $\mathcal{S}(\mathbb{R})$, since e^{ikx} & $\delta(x-a)$ aren't in $\mathcal{S}(\mathbb{R})$.)

Here's <u>one</u> eigenvector of H:

$$\psi_0(x) = e^{-x^2/2}$$

Let's check how this works:

$$\frac{d\psi_0}{dx} = -x e^{-x^2/2}$$

$$\frac{d^2\psi_0}{dx^2} = -(1+x^2)e^{-x^2/2}$$

$$p^2\psi_0 = (1-x^2)\psi_0$$

$$q^2\psi_0 = x^2\psi_0$$

$$\Rightarrow H\psi_0 = \frac{1}{2}(p^2+q^2)\psi_0 = \frac{1}{2}\psi_0.$$

It turns out $\frac{1}{2}$ his the lowest eigenvalue of H. In general, if

$$H\psi = E\psi$$

we say ψ describes a state where the particle has energy E.

So $\psi_0 = e^{-x^2/2}$ is the state of least energy, i.e. the ground state of the harmonic oscillator. Note that the ground state energy is $\frac{1}{2}$, not (as in the classical harmonic oscillator) 0!

really $\frac{1}{2}\hbar\omega$

Due to the uncertainty principle we can't get both p & q to be 0 in QM. Now how do we get more eigenvectors?

We'll now define cts. lin. ops

$$a, a^* : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

s.t. if

$$H\psi = E\psi$$

LADDER OPERATORS
aka
RAISING and LOWERING OPS
aka
CREATION / ANNIHILATION OPS.

then

$$H a^*\psi = (E+1) a^*\psi$$

(so a^* raises the energy by 1)

&

$$H a\psi = (E-1) a\psi$$

(a lowers the energy by 1)

We call a the <u>annihilation operator</u> (because $a\psi_0 = 0$ – a annihilates ψ_0) or <u>lowering operator</u>; we call a^* the <u>creation op.</u> or <u>raising op.</u>

← note from the def. of a, we must have $a\psi_0 = 0$, or else ψ_0 couldn't be the ground state.

These ladder operators are

$$a = q + ip$$
$$a^* = q - ip$$

(People usually divide these by $\sqrt{2}$; we'll do that later.)

Recall:

$$\frac{dO(t)}{dt} = i\,[H, O(t)]$$

$$[iH, p] = -q \qquad [iH, q] = p$$

so

$$\begin{aligned}[iH, a] &= [iH, q + ip] \\ &= [iH, q] + [iH, ip] \\ &= p - iq \\ &= -ia\end{aligned}$$

$$\Rightarrow \boxed{[H, a] = -a}$$

$$\begin{aligned}[iH, a^*] &= [iH, q - ip] \\ &= p + iq \\ &= ia^*\end{aligned}$$

$$\Rightarrow \boxed{[H, a^*] = a^*}$$

So if $H\psi = E\psi$ then

$$\begin{aligned}Ha^*\psi &= (a^*H + [H, a^*])\,\psi \\ &= (a^*H + a^*)\,\psi \\ &= a^*E + a^* \\ &= (E+1)\,a^*\psi\end{aligned}$$

&

$$\begin{aligned}Ha\psi &= (aH + [H, a])\,\psi \\ &= (aE - a)\,\psi \\ &= (E-1)\,a\psi\end{aligned}$$

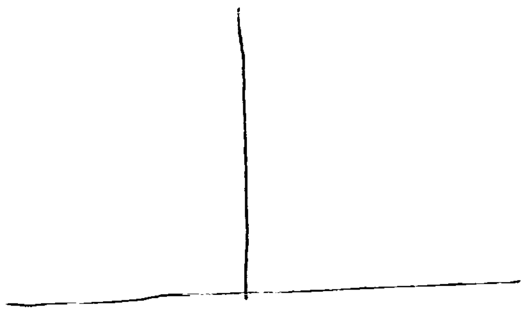
So we can take ψ_0 and hit it with a^* n times & get

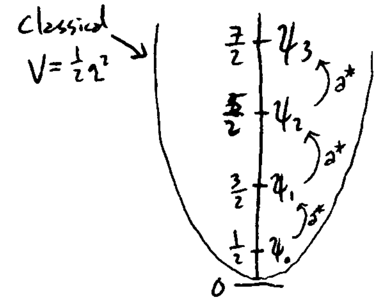
$$\psi_n = a^{*n}\psi_0$$

so

$$H\psi_n = (n + \tfrac{1}{2})\,\psi_n .$$

These are in fact <u>all</u> of the eigenvectors of H, so we have:





What happens if we apply the lowering operator a to ψ_n?

$$\begin{aligned} a\psi_0 &= (q+ip)\psi_0 \\ &= \left(x+\frac{d}{dx}\right)e^{-x^2/2} \\ &= 0 , \quad \text{as expected.} \end{aligned}$$

If $n \geq 1$,

$$a\psi_n = a\, a^{*n}\psi_0$$

To do this we would like to commute the a through all of the a^*'s. We need the commutator:

$$\begin{aligned} [a, a^*] &= [q+ip, q-ip] \\ &= [q,q] + i[p,q] - i[q,p] + [p,p] \\ &= 2i[p,q] \\ &= 2i(-i) \\ &= 2 \end{aligned}$$

this 2 is why people usually divide and a^* by $\sqrt{2}$.

So:

$$\begin{aligned} a\psi_n &= a a^* a^{*n-1}\psi_0 \\ &= (a^*a + [a,a^*])\, a^{*n-1}\psi_0 \\ &= (a^*a + 2)\, a^{*n-1}\psi_0 = \cdots \end{aligned}$$

& etc. pushing a to the right.

An easier way:

$[a, \cdot]$ is a derivation, so

$[a, a^*] = 2$ implies

$[a, \cdot] = $ "$2\frac{d}{da^*}$"

$$\boxed{[a, a^{*n}] = 2n a^{*n-1}}$$

Then

$$a\psi_n = a\, a^{*n}\psi_0 = (a^{*n}a + [a, a^{*n}])\psi_0$$
$$= (0 + 2na^{*n-1})\psi_0$$
$$= 2n\,\psi_{n-1}$$

Note: a & a^* are not inverses of each other.

There's a big difference between creating a particle and then annihilating a particle, and the other way round.

Usually, people get rid of these "2's" by setting

$$a = \frac{q + ip}{\sqrt{2}} \qquad a^* = \frac{q - ip}{\sqrt{2}}$$

Then

$$[a, a^*] = 1$$

but we still have $[H, a] = -a$

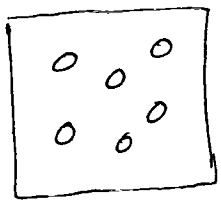
$[H, a^*] = a^*$

& now if we set

$$\psi_n = a^{*n}\psi_0$$

we get

$$H\psi_n = (n+\tfrac{1}{2})\psi_n$$
$$a^*\psi_n = \psi_{n+1}$$
$$a\psi_n = n\psi_{n-1}$$



ψ_n is like a box
with n indistinguishable balls

$a^*\psi_n = \psi_{n+1}$ says there's only one way to add a ball to the box.

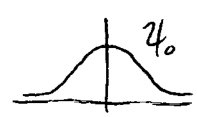
$a\psi_n = n\psi$ says there are n ways to take a ball out.

$$[a, a^*] = 1 \quad \text{or}$$
$$a a^* = a^* a + 1$$

"There's one more way to ~~take~~ put in a ball and then take one out than to take one out and then put one in.

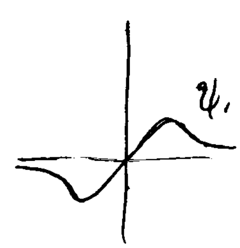
What are the functions ψ_n like?

$$\psi_0 = e^{-x^2/2}$$

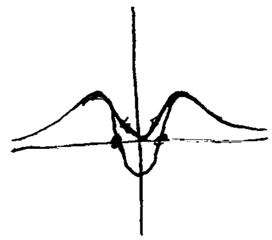


the gaussian of std. deviation 1.

$$\psi_1 = a^*\psi_0 = \frac{q-ip}{\sqrt{2}}\psi_0$$
$$= \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)e^{-x^2/2}$$
$$= \frac{1}{\sqrt{2}}\, 2x\, e^{-x^2/2}$$



$$\begin{aligned}
\psi_2 &= a^* \psi_1 \\
&= \tfrac{1}{\cancel{2}} \left(x - \tfrac{d}{dx}\right) \cancel{2} x e^{-x^2/2} \\
&= (x^2 - 1 + x^2) e^{-x^2/2} \\
&= (2x^2 - 1) e^{-x^2/2}
\end{aligned}$$



Certainly

$$\psi_n(x) = H_n(x) e^{-x^2/2}$$

where H_n is some polynomial of degree n – the nth <u>Hermite polynomial</u> (up to fudge factors). It's easy to see $H_n(x)$ has degree n, and H_n is also even/odd depending on parity of n (this comes from the fact that mult by x and diff by x both turn odd fns into even fns and even fns into odd fns, so $x - \frac{d}{dx}$ does too). It's also true (but not trivial) that H_n has n real roots.

<u>Thm</u>: The fns. ψ_n form a (topological) basis of $\mathcal{S}(\mathbb{R})$ i.e. they're linearly indep. & the finite lin. combs $\sum_{n=0}^{N} a_n \psi_n$ are dense in $\mathcal{S}(\mathbb{R})$ ← A souped up version of the Stone Weierstrass Thm but noncompactness makes it harder.

"<u>Pf</u>:" They're obviously lin. indep. & finite lin. combs are the same as fns $P(x)e^{-x^2/2}$ where P is any poly. So we need: polynomials times $e^{-x^2/2}$ are dense in $\mathcal{S}(\mathbb{R})$. This follows from the "L^2 Stone-Weierstrass Thm" & some extra stuff. ∎

Prop. - ψ_n are orthogonal in $L^2(\mathbb{R})$:

$$\langle \psi_n, \psi_m \rangle = \int_{-\infty}^{\infty} \overline{\psi_n}\, \psi_m \, dx = 0 \quad \text{if } n \neq m$$

Pf: Note

$$\langle p\psi, \varphi \rangle = \langle \psi, p\varphi \rangle$$

$\forall \psi, \varphi \in \mathcal{S}(\mathbb{R})$

$$\langle q\psi, \varphi \rangle = \langle \psi, q\varphi \rangle$$

via integration by parts (note this works rigorously: $\int_{-\infty}^{\infty} \overline{-i\psi'}\, \varphi = \int_{-\infty}^{\infty} \overline{\psi} \cdot -i\varphi'$ + boundary terms which are zero because our functions go to zero fast enough.)

$$\int x \overline{\psi} \varphi = \int \overline{\psi}\, x\varphi$$

So

$$\langle H\psi, \varphi \rangle = \langle \psi, H\varphi \rangle \quad \text{since } H = \tfrac{1}{2}(p^2 + q^2)$$

Then

$$\frac{1}{E_n} \langle H\psi_n, \psi_m \rangle = \langle \psi_n, \psi_m \rangle = \frac{1}{E_m} \langle \psi_n, H\psi_m \rangle$$

$$\parallel$$

$$\frac{1}{E_n} \langle \psi_n, H\psi_m \rangle$$

So either $E_n = E_m$ or $\langle \psi_n, \psi_m \rangle = 0$. ∎

(note: $E_n \neq 0$)

Prop: $\mathcal{S}(\mathbb{R})$ are dense in $L^2(\mathbb{R})$.

Pf: Even $C_0^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ are dense in $L^2(\mathbb{R})$. ∎

<u>Thm:</u> If we normalize ψ_n to get

$$|n\rangle = \frac{\psi_n}{\|\psi_n\|}$$

these form an o.n. basis of $L^2(\mathbb{R})$.

<u>Pf:</u> They're orthonormal, since the ψ_n are orthogonal. They form a basis since the finite lin. combs. of the ψ_n are dense in $\mathcal{S}(\mathbb{R})$ & thus by previous prop. in $L^2(\mathbb{R})$. ∎

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21 October 2003

The Weyl Algebra: the assoc. alg. gen by p, q s.t.

$$[p,q] = -i.$$

is also generated by

$$a = \frac{q+ip}{\sqrt{2}}$$

$$a^* = \frac{q-ip}{\sqrt{2}}$$

with $[a,a^*] = 1$, since

$$\frac{a+a^*}{\sqrt{2}} = q$$

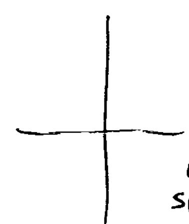
$$\frac{a-a^*}{\sqrt{2}\,i} = p$$

note the similarity with

$z = x+iy$
$\bar{z} = x-iy$

$\frac{z+\bar{z}}{2} = x$
& $\frac{z-\bar{z}}{2i} = y$

$\mathbb{R}^2 \cong \mathbb{C}$



is phase space for classical harm. oscillator

We considered the <u>Schrödinger rep</u> of the Weyl algebra, where

$$p, q : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

$$\mathcal{S}(\mathbb{R}) = \left\{ \psi : \mathbb{R} \to \mathbb{C} \; : \; \left| x^n \frac{d^n}{dx^n} \psi \right| \text{ bded} \right\}$$

Now we're doing a quantum version of the same thing: a, a^* are like z & $\bar{z}$ (→ noncommutative complex analysis)