There's a rig homomorphism

$$f : \mathbb{N} \longrightarrow \Omega$$
$$0 \longmapsto F$$
$$1, 2, 3, \ldots \longmapsto T$$

and this introduces a rig homo. $M_n(\mathbb{N}) \to M_n(\Omega)$

Going from $\mathbb{N}$ to $\mathbb{C}$ is still a big jump!

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7 October 2003

Matrix Mechanics

Rigs R and their matrix rigs $M_n(R)$

0.) No rigs w. zero elts.

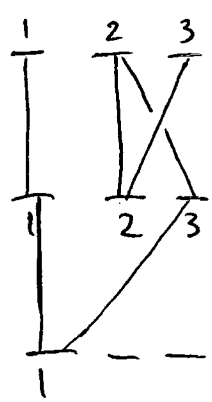
1.) One rig w. one elt. There's at most one rig homo from any rig R to this one, since there's only one map to the one-elt. set, and this indeed is a rig homo, so there's exactly one rig homo from any rig to this one! So we say this is the terminal rig, and call this rig 1.

2.) Two rigs w. 2 elts:

$$\mathbb{Z}_2 \;\&\; \Omega = \{\{F,T\}, \vee, \wedge, F, T\}$$

We saw that elts of $M_n(\Omega)$ are binary relations on the set $\{1, \ldots, n\}$

composition of processes (composition of relations) = matrix multiplication.



The entries A_{ij} of $A \in M(\Omega)$ say whether or not the process A can take the state i to the state j.

Matrix multiplication is "composition of processes"; addition is "superposition" of process

Note there's no destructive interference since Ω is a rig but not a ring. in $\mathbb{Z}_2$ there is destructive interference.

3.) $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ There's at most one rig homo from $\mathbb{N}$ to any other rig, since $f: \mathbb{N} \to R$ must have $f(0) = 0$, $f(1) = 1$ & thus $f(n) = \underbrace{f(1) + \cdots + f(1)}_{n}$. This also lets us define f; check that f really is a rig homo. So we say $\mathbb{N}$ is <u>initial</u>.

The entries of $A_{ij} \in M_n(\mathbb{N})$ say how many ways the process A can take state j to state i.

4.) $\mathbb{R}^+ = ([0, \infty), +, \cdot, 0, 1)$.

The entries of $A_{ij} \in M_n(\mathbb{R}^+)$ say what's the relative probability that process takes state j to state i.

This is nicest if the matrix is <u>stochastic</u>:

$$\sum_i A_{ij} = 1$$

– the prob. that state j goes <u>somewhere</u> is 1.

A, B stochastic $\Longrightarrow$ AB stochastic.

$\psi \in (\mathbb{R}^+)^n$ describes the relative probabilities for the events $i = 1, \ldots, n$ to occur. We say ψ is <u>normalized</u> if

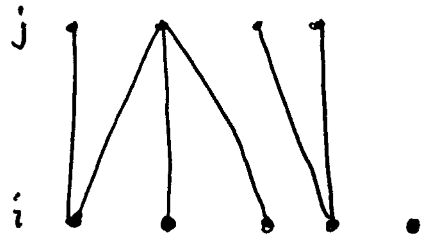
$$\sum \psi_i = 1.$$

In this case the rel. probs. ψ_i are actual probabilities. We can normalize any nonzero ψ.

Given a normalized ψ and a stochastic A, then $A\psi$ is normalized:

$$\sum_i (A\psi)_i = \sum_{i,j} A_{ij} \psi_j = \sum_j \psi_j = 1.$$

If we go back to the rig Ω of truth values, then the analog of a stochastic matrix is a total relation: one s.t. every j is related to some i (but may be more than one. e.g.



A vector $\psi \in \Omega^n$ is a subset of n, & $\bigvee_i \psi_i = T$ says the subset is nonempty.

A silly aside.

Thm: $\sqrt[n]{2}$ is irrational $\forall n > 2$

Pf: by contradiction:

$$\sqrt[n]{2} = \frac{p}{q}$$

$$2 = \frac{p^n}{q^n}$$

$$q^n + q^n = p^n$$

By Fermat's last theorem, this is a contradiction!

For $\mathbb{N}$, a stochastic matrix becomes a function $f: \{1, \ldots, n\} \to \{1, \ldots, n\}$ since $\forall j \in \{1, \ldots, n\}$ $\sum_i A_{ij} = 1$ so $\exists!$ i st. $A_{ij} = 1$, so can $i = f(j)$ for some function i. $\psi \in \mathbb{N}^n$ w. $\sum \psi_i = 1$ determines an element of $\{1, \ldots, n\}$.

"Sets & Functions is matrix mechanics over $\mathbb{N}$."

 In QM we usually use:

6) $\mathbb{C} = (\mathbb{C}, +, \cdot, 0, 1)$.

If $A \in M_n(\mathbb{C})$ then A_{ij} says the <u>relative amplitude</u> to get from state j to state i via process A.

We'll say $\psi \in \mathbb{C}^n$ is normalized if

$$\sum |\psi_i|^2 = 1$$

- then $|\psi_i|^2$ says the probability for the event i to occur.

We say $A \in M_n(\mathbb{C})$ is <u>unitary</u> or

$$\sum_j |A_{ij}|^2 = 1$$

ψ normalized, A unitary $\Longrightarrow A\psi$ normalized.

In QM, a normalized vector in $\mathbb{C}^n$ (or complex Hilbert space) is called:

- state vector
- state
- wavefunction.

<u>Note</u>: $\sum |\psi_i|^2 = \sum \psi_i^* \psi_i$ uses the fact that $\mathbb{C}$ is a $*$-rig, i.e. a rig R w. $*: R \to R$ s.t. $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$, $0^* = 0$, $1^* = 1$. Any commutative rig is a $*$-rig with $* = \mathrm{id}$, so we could go back & use $\sum \psi_i^* \psi_i$, $\sum A_{ij} A_{ij}^*$ in previous examples. (This makes $\mathbb{C}$ look less like an exception)

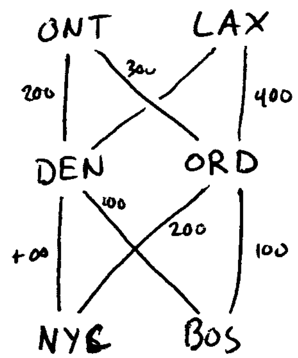
(Exercise: do this.)

7) $K = \left((-\infty, +\infty],\ \overset{\text{"add"}}{\min},\ \overset{\text{"mult."}}{+},\ +\infty,\ 0\right)$

is $\begin{cases} \text{Jim Dolan's} \\ \text{or Bellman's} \end{cases}$ rig of <u>costs.</u> Given $A \in M_n(K)$,

Jim said he attributes it to Bellman

A_{ij} says the <u>cost</u> of going from j to i.



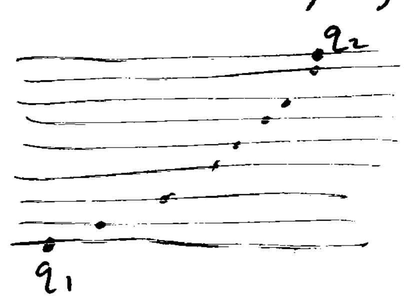
Cost of going from LAX to BOS $= (200+100) \min (400+100)$ ~~Th.~~ – a sum of products in the rig K. To compute the cost of a composite process, we do matrix mult. in $M_n(K)$.

In <u>Classical Mechanics</u>, ~~particles travel along~~ (in one dimension) a particle going from position q_1 at time t_1 to position q_2 at t_2 usually takes the path

$$q : [t_1, t_2] \longrightarrow \mathbb{R}$$

that minimizes the <u>action</u>

$$S(q) = \int_{t_1}^{t_2} \underbrace{K(\dot{q}(t)) - V(q(t))}_{\text{Lagrangian.}}\, dt$$



We can solve this by discretizing time, multiplying a bunch of $\mathbb{R}\times\mathbb{R}$ matrices valued in K, then taking limit as $\Delta t \to 0$.

This lets us turn Lagrangian approach to CM into matrix mechanics over the rig of costs (or rather, <u>actions</u>) "Matrix mechanics over K."

There's also a Lagrangian approach to QM which is related to matrix mechanics over $\mathbb{C}$.

<u>October 9, 2003</u>

One more "rig":

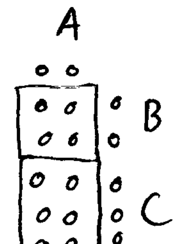
The Category FinSet. It's like a rig with

$+$ = disjoint union
$\cdot$ = Cartesian product
0 = $\emptyset$
1 = 1-elt set.

All the rig axioms hold up to canonical isomorphism, e.g.

$$A\times(B+C) \cong A\times B + A\times C$$

<u>example:</u>



recall underline means decategorification

Since $\underline{\text{FinSet}} = \mathbb{N}$, all these operations on FinSet give ops on $\mathbb{N}$ making $\mathbb{N}$ into a rig.

FinSet is a "categorified rig" or "2-rig". Just as $\mathbb{N}$ is the <u>initial</u> rig., FinSet is the "initial 2-rig" (the free 2-rig on nothing (Just as $\mathbb{N}$ is the free rig on nothing)).

Aside:
Example: what's the free rig on 1 generator?
Form all possible elts using no equations but those in the definition of a rig.
e.g. $x \quad (x+x)+x = x+(x+x), \ x^2, \ x^2+x, \ldots$
so it's $\mathbb{N}[x]$.

Plan: (1) Study harmonic oscillator & see that the Hilbert space of states is (a completion of) $\mathbb{C}[x]$.

(2) Notice we can just use $\mathbb{N}$ instead of $\mathbb{C}$ & get "natural numbers oscillator" w. $\mathbb{N}[x]$ as its space of states. (or maybe some completion of $\mathbb{N}[x]$).

(3) $\mathbb{N}[x]$ is the free rig on one generator; why not categorify and use the free 2-rig on one generator, FinSet$[x]$ — also called the category of <u>species</u>, invented by André Joyal to study combinatorics.

Upshot: "categorifying the (natural numbers) harmonic oscillator gives combinatorics."

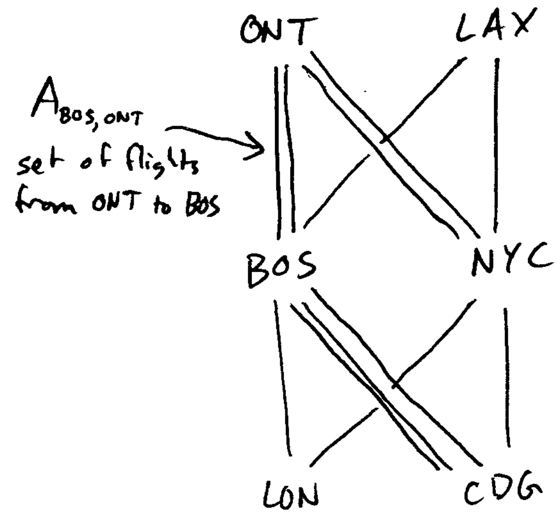
~~Back to Heisenberg's matrix mechanics~~

We can let $M_n(\mathrm{FinSet})$ be the category whose objects are $n \times n$ matrices of finite sets, & whose morphisms are $n \times n$ matrices of functions

$$\begin{pmatrix} f & g \\ h & i \end{pmatrix} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \qquad \text{where} \quad \begin{array}{l} f : A \to A' \\ g : B \to B' \\ h : C \to C' \\ i : D \to D' \end{array}$$

with obvious composition & identities.

Given a matrix $A \in M_n(\mathrm{FinSet})$, the entry A_{ij} describes the finite set of ways of getting from state j to state i.



We describe composition of these processes by matrix multiplication in $M_n(\mathrm{FinSet})$:

$$(AB)_{ik} = \underbrace{\sum_j}_{\text{disjoint union}} A_{ij} \underbrace{\times}_{\text{cartesian product}} B_{jk}$$

Now... Back to Heisenberg's Matrix Mechanics, for the particle on a line, especially the harmonic oscillator.

Recall: classically observables for particle on line form the comm. alg. $C^\infty(\mathbb{R}^2)$ where $(p,q) \in \mathbb{R}^2$, which becomes a Poisson algebra with

$$\{p,q\} = 1 \qquad \begin{array}{l} \{p,p\} = 0 \\ \{q,q\} = 0 \end{array}$$

& if the Hamiltonian $H \in C^\infty(\mathbb{R}^2)$ is

$$H(p,q) = \frac{p^2}{2} + V(q) \qquad \begin{array}{c} (m=1) \\ V \in C^\infty(\mathbb{R}) \end{array}$$

then any observable O gives a 1-parameter family of observables $O(t)$ s.t.

$$\frac{d}{dt} O(t) = \{H, O(t)\}$$

(this is general, independent of form of Hamiltonian) & in particular for <u>this</u> Hamiltonian:

$$\frac{d}{dt} q(t) = p(t)$$

$$\frac{d}{dt} p(t) = -V'(q).$$

For the harmonic oscillator,

$$V(x) = \frac{1}{2}x^2$$

these reduce to

$$\frac{dq(t)}{dt} = p(t) \qquad \frac{dp(t)}{dt} = -q(t)$$

In Heisenberg's approach to quantum mechanics, observables form a noncomm. algebra containing elts p & q s.t.

$$[p,q] = -i\hbar$$

& if the Hamiltonian is some observable

$$H = \frac{p^2}{2} + V(q)$$

(easy to define if V is algebraic (a polynomial), harder otherwise - use the functional calculus)

then any observable O gives observables $O(t)$ s.t.

$$-i\hbar \frac{d}{dt} O(t) = [H, O(t)]$$

[(Problem: does this d.e. have a solution? In the classical case, the answer has to do with integrability of vector fields (see JB's CM notes))]

Really? Is there a solution?

The solution wants to be:

$$O(t) = e^{itH/\hbar} O e^{-itH/\hbar}$$

because:

$$\frac{d}{dt} O(t) = \frac{d}{dt}\left(e^{itH/\hbar} O e^{-itH/\hbar}\right)$$

$$= \frac{d}{dt}\left(e^{itH/\hbar}\right) O e^{-itH/\hbar} + e^{itH/\hbar} O \frac{d}{dt}\left(e^{-itH/\hbar}\right)$$

$$= \frac{iH}{\hbar} e^{itH/\hbar} O e^{-itH/\hbar} - e^{itH/\hbar} O e^{itH/\hbar} \frac{iH}{\hbar}$$

can pull out H on either side because H commutes with $e^{itH/\hbar} = f(H)$

Functional Calculus!

$$= \frac{i}{\hbar}\left[H, O(t)\right]$$

as desired

Note: $\hbar$ has units of action = time × energy, so $\frac{tH}{\hbar}$ is dimensionless so $e^{itH/\hbar}$ makes sense & is dimensionless. Pick units of action so $\hbar = 1$ now! Then the new rules of the game are:

$$[p, q] = -i$$

$$-i\frac{dO(t)}{dt} = [H, O(t)]$$

This should have the solution:

$$O(t) = e^{itH} O e^{-itH}$$

<u>Problem:</u> $e^{itH} = \sum_{n \geq 0} \frac{(itH)^n}{n!}$ doesn't lie in alg. gen by p & q - even if V is a polynomial. So, we need some completion of this algebra.

Now let's start calculating

$$[p, q] = -i$$

To do $[p, q^2] = ?$ you use:

$$[A, BC] = [A,B]C + B[A,C]$$

$$ABC - BCA = ABC - BAC + BAC - BCA$$

(so $[A, -]$ is a derivation, just as $\{A, \cdot\}$ was a derivation of the comm. alg. in classical mechanics)

$$[p, q^2] = [p,q]q + q[p,q] = -2iq$$

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14 October 2003

JB forgot his notebook today...

The Weyl algebra is the associative alg. over $\mathbb{C}$ generated by p, q satisfying

$$pq - qp = -i$$

We'll do calculations in this algebra (or some larger algebra containing things like e^{itH} where H is an elt. of Weyl algebra). Starting with:

$$[p,q] = -i$$

$$[p,q^2] = [p,q]q + q[p,q] = -2iq$$

$$\vdots$$

$$[p,q^n] = -inq^{n-1}$$

or for any polynomial F,

$$[p, F(q)] = -iF'(q)$$

o.k. for short

$$[p, \cdot] = -i\frac{\partial}{\partial q}.$$

Next:

$$[q,p^2] = [q,p]p + p[q,p] = 2ip$$

$$[q,p^n] = inp^{n-1}$$

or for any poly. F

$$[q, F(p)] = iF'(p)$$

or:

$$[q, \cdot] = i\frac{\partial}{\partial p}.$$