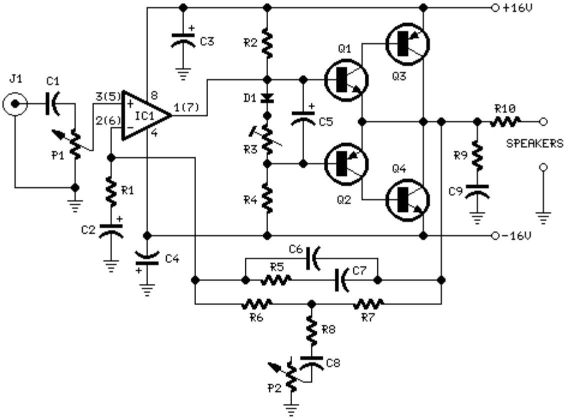
CIRCUITS, CATEGORIES AND REWRITE RULES



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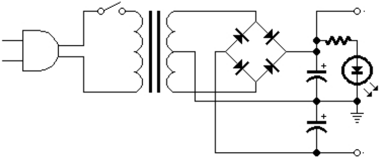
Higher-Dimensional Rewriting and Applications

Warsaw, 29 June 2015

If mathematicians want to understand networks, a good place to start is electrical circuits.

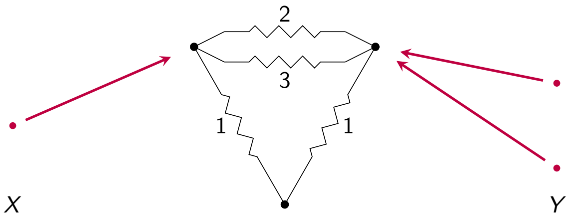


They look like string diagrams for morphisms in symmetric monoidal categories... *and they are!* Their mathematics is well-studied, but not yet fully formalized. The modern world is *made of them*.

To keep things simple, today let's just talk about circuits made of resistors — a special case of my work with Brendan Fong.

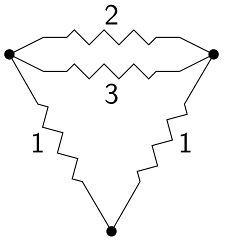
We will:

- Describe a category **Circ** where morphisms are electrical circuits made out of resistors.



- Describe the ‘black box functor’ $\blacksquare\colon \mathbf{Circ} \to \mathbf{FinRel}_{\mathbb{R}}$ sending each circuit to its ‘behavior’: the linear relation it imposes between electric potentials and currents at its inputs and outputs.
- Study rewrite rules that go between circuits with the same behavior.

For today, define a **circuit** to be a graph with edges labelled by positive real numbers called **resistances**:



More precisely, we have:

- a directed graph $s, t\colon E \to N$ with a set of **edges** E and a set of **nodes** N, each edge $e \in E$ having a **source** $s(e) \in N$ and **target** $t(e) \in N$,
- a **resistance** for each edge: $r\colon E \to (0, \infty)$.

When we use it, a circuit has:

- a **potential** $\phi(n) \in \mathbb{R}$ at each node $n \in N$:

$$\phi \colon N \to \mathbb{R}$$

- a **current** $I(e) \in \mathbb{R}$ along each edge $e \in E$:

$$I \colon E \to \mathbb{R}$$

These obey some linear relations, which is how we get circuits to do interesting things for us.

If we fix the potential at some set of **terminals** $T \subseteq N$, the circuit automatically chooses potentials at all the other nodes, and currents along all the edges. These obey:

- **Ohm's law**: the current along an edge is proportional to the change in potential as we go from its source to its target:

$$I(e) = \frac{\phi(s(e)) - \phi(t(e))}{r(e)}$$

- **Kirchhoff's current law**: the total current flowing out of a node $n \in N - T$ equals the total current flowing in:

$$\sum_{e \text{ such that } s(e)=n} I(e) \quad = \sum_{e \text{ such that } t(e)=n} I(e)$$

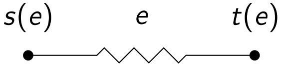
At the terminals there can be a net inflow or outflow of current!

That is, if we define the **current flowing out of the node** n by:

$$I(n) = \sum_{e \text{ such that } s(e)=n} I(e) \quad - \sum_{e \text{ such that } t(e)=n} I(e)$$

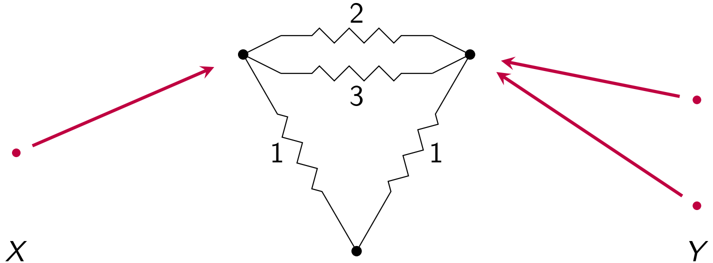
then $I(n) = 0$ if $n \in N - T$, but $I(n)$ can be nonzero if $n \in T$.

For example, this simple circuit with two terminals can have current flowing into $s(e)$ and out of $t(e)$:



$$I(e) = \frac{\phi(s(e)) - \phi(t(e))}{r(e)}$$

Define an **open circuit** to be a circuit with chosen **inputs** and **outputs**:



picked out by maps

$$i\colon X \to N$$

$$o\colon Y \to N$$

Here the terminals are $i(X) \cup o(Y)$.

Thanks to Ohm's law and Kirchhoff's current law, an open circuit imposes linear equations relating:

- the potentials at the inputs and currents flowing *into* the inputs:

$$\phi(i(x)),\ -I(i(x)) \qquad \text{for } x \in X$$

and:

- the potentials at the inputs and currents flowing *out of* the outputs:

$$\phi(o(y)),\ I(o(y)) \qquad \text{for } y \in Y$$

Thus, it picks out a linear subspace

$$L \subseteq (\mathbb{R}^X \oplus \mathbb{R}^X) \oplus (\mathbb{R}^Y \oplus \mathbb{R}^Y)$$

describing the allowed potentials and currents at inputs and outputs.

The linear subspace

$$L \subseteq (\mathbb{R}^X \oplus \mathbb{R}^X) \oplus (\mathbb{R}^Y \oplus \mathbb{R}^Y)$$

defined by an open circuit with inputs X and outputs Y is a morphism

$$L \colon \mathbb{R}^X \oplus \mathbb{R}^X \rightsquigarrow \mathbb{R}^Y \oplus \mathbb{R}^Y$$

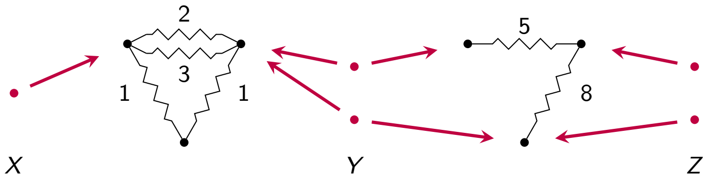
in the category $\mathbf{FinRel}_{\mathbb{R}}$ of finite-dimensional real vector spaces and linear relations. We call this the circuit's **behavior**. It says what the circuit does 'as seen from outside'.

This suggests finding a category **Circ** whose morphisms are open circuits, and a functor

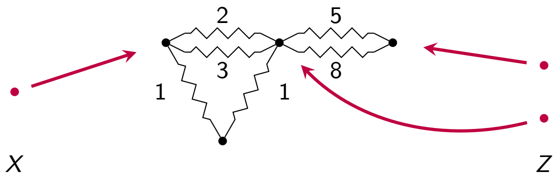
$$\blacksquare \colon \mathbf{Circ} \to \mathbf{FinRel}_{\mathbb{R}}$$

mapping each open circuit to its behavior.

We can compose open circuits by gluing the outputs of one to the inputs of another. These open circuits:



compose to give this:



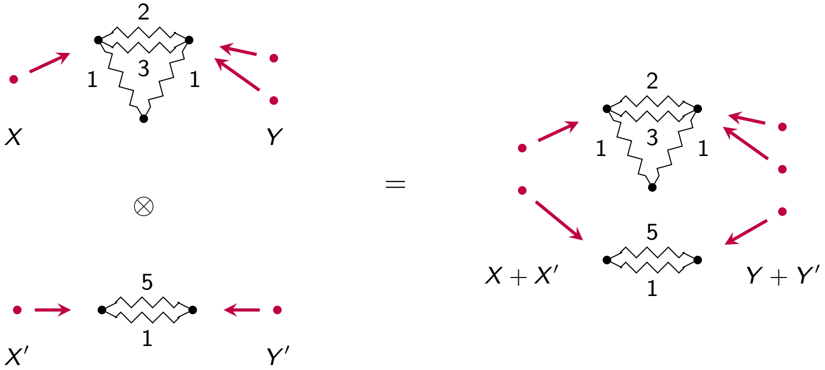
Indeed, there is a category **Circ** where:

- objects are finite sets
- a morphism $f \colon X \to Y$ is an isomorphism class of open circuits from X to Y. An **open circuit from $\boldsymbol{X}$ to $\boldsymbol{Y}$** is a cospan of finite sets

$$X \xrightarrow{\;i\;} N \xleftarrow{\;o\;} Y$$

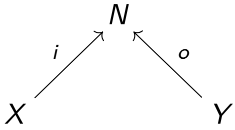
where N is the nodes of a graph $s, t \colon E \to N$ with edges labelled by resistances via $r \colon E \to (0, \infty)$.

We can also 'tensor' open circuits:

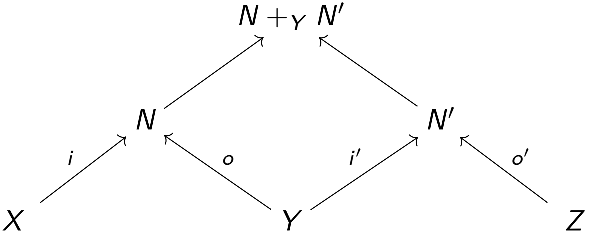


How do we prove **Circ** is a symmetric monoidal category? We can use Fong's theory of decorated cospans.

Say we start with a category **C** with finite colimits: in our example, **C** = **FinSet**. We can make a bicategory where morphisms are cospans in **C**:

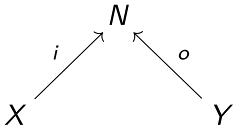


and composition is done by pushout:



The pushout is defined only up to isomorphism, but there is a category **Cospan**(**C**) where morphisms are *isomorphism classes* of cospans in **C**.

If we choose a functor $F\colon \mathbf{C} \to \mathbf{D}$, we can try to create a category where a morphism is an isomorphism class of cospans



with N 'decorated' by an element of $F(N)$. In our example, $\mathbf{D} = \mathbf{Set}$ and $F(N)$ is the set of ways of making N into the nodes of a circuit.

But what's an 'element', in general? And how do we 'compose the decorations' when we compose cospans?

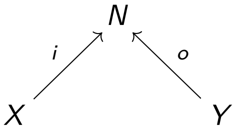
Theorem (Fong)

Suppose **C** *has finite colimits,* **D** *is symmetric monoidal, and*

$$F\colon (\mathbf{C}, +) \longrightarrow (\mathbf{D}, \otimes)$$

is a lax symmetric monoidal functor. Then there is a symmetric monoidal category of ***F*-decorated cospans**, ***F*Cospan**, *where:*

- *an object is an object of* **C**,
- *a morphism from X to Y is a cospan*



together with a **decoration** $r\colon 1 \to F(N)$. *(Just kidding: actually, a morphism is an* isomorphism class *of these!)*

Given composable morphisms

$$X \xrightarrow{i} N \xleftarrow{o} Y \qquad r\colon 1 \to F(N)$$

$$Y \xrightarrow{i} N' \xleftarrow{o} Z \qquad r'\colon 1 \to F(N')$$

we compose the cospans by taking the pushout, and compose the decorations by forming

$$1 \xrightarrow{r \otimes r'} F(N) \otimes F(N') \longrightarrow F(N + N') \longrightarrow F(N +_Y N')$$

where the second arrow comes from F being a *lax* monoidal functor.

So, let $F\colon (\mathbf{FinSet}, +) \to (\mathbf{Set}, \times)$ be the lax monoidal functor sending any finite set N to the set of ways of making N into the nodes of a circuit. Then $F\mathbf{Cospan} = \mathbf{Circ}$ is a symmetric monoidal category whose morphisms are open circuits.

This lays the groundwork for the main result:

Theorem (Fong)

There is a symmetric monoidal functor

$$\blacksquare\colon \mathbf{Circ} \to \mathbf{FinRel}_{\mathbb{R}}$$

mapping any finite set X to the vector space $\mathbb{R}^X \oplus \mathbb{R}^X$, and mapping any open circuit to its behavior.

We can ask various questions about the black box functor, both in our example and for other kinds of circuits:

- What morphisms are in the image of $\blacksquare$? That is: which behaviors can be realized by some circuit?
- Given a morphism in the image of $\blacksquare$, how can we find a circuit that gives this morphism? Is there a 'simplest' one?
- How can we tell if two circuits have the same behavior? Can we find *rewrite rules* that take us between any two circuits with the same behavior? Can we use these to 'simplify' any circuit until we reach a 'normal form'?

The first question is addressed by our paper. The second has been studied in detail for 'planar' circuits.

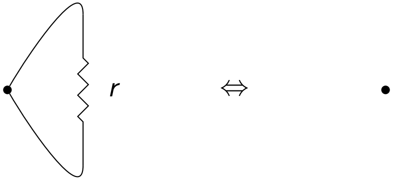
There is a category **PlanarCirc** where morphisms come from 'planar' open circuits: roughly, open circuits drawn on a square with inputs on top and outputs on bottom. There is again a black box functor

$$\blacksquare \colon \mathbf{PlanarCirc} \to \mathbf{FinRel}_{\mathbb{R}}$$

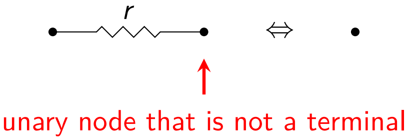
In this case, repeated use of five rewrite rules is enough to carry us from any morphism f to any other morphism g with

$$\blacksquare(f) = \blacksquare(g)$$

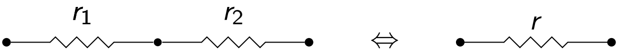
1. A **self-loop** can be deleted:



2. A **spike** can be deleted. That is: a unary node that is not a terminal can be deleted along with its incident edge:



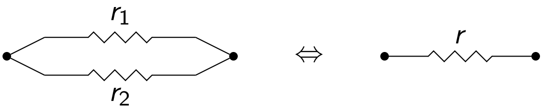
3. Resistors in **series** can be replaced by a single resistor:



where

$$r = r_1 + r_2$$

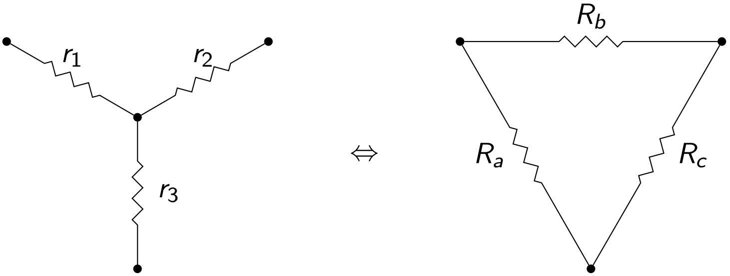
4. Resistors in **parallel** can be replaced by a single resistor:



where

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

5. The **Y-Δ transform**:



where

$$r_1 = \frac{R_a R_b}{R_a + R_b + R_c}$$

$$r_2 = \frac{R_b R_c}{R_a + R_b + R_c}$$

$$r_3 = \frac{R_c R_a}{R_a + R_b + R_c}$$

We *cannot* choose directions on these rewrite rules to make them terminating and confluent. So, we don't obtain a 'normal form'.

But there is for each morphism a family of 'best possible forms', and these put a rich mathematical structure on the image of the black box functor!

For details, see:

- Colin de Verdière, Réseaux électriques planaires I.
- Colin de Verdière, Gitler and Vertigan, Réseaux électriques planaires II.
- Curtis, Ingerman and Morrow, Circular planar graphs and resistor networks.
- Alman, Lian and Tran, Circular planar electrical networks I: The electrical poset EP_n.
- Alman, Lian and Tran, Circular planar electrical networks II: Positivity phenomena.