Higher Categories, Higher Gauge Theory – II

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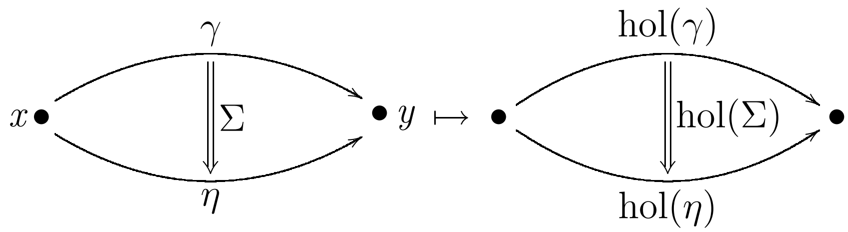


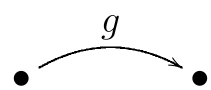
Notes and references at:

http://math.ucr.edu/home/baez/namboodiri/

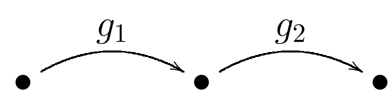
Gauge Theory

Ordinary gauge theory describes how point particles transform as they move along paths in spacetime:

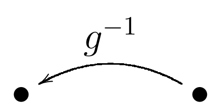
It's natural to assign a *group* element to each path, called its 'holonomy':



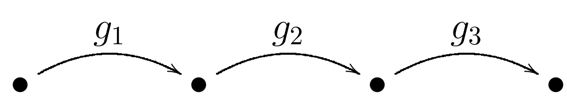
and require that composing paths correspond to multiplying holonomies:



while reversing a path corresponds to taking the inverse of its holonomy:



The associative law makes the holonomy along a triple composite unambiguous:

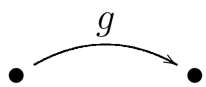


In short: *the topology dictates the algebra!*

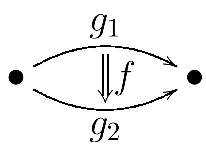
The electromagnetic field is described using the group U(1). Other forces are described using other groups.

Higher Gauge Theory

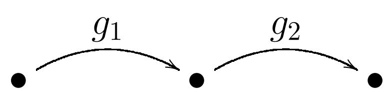
Higher gauge theory describes not just how point particles but also how 1-dimensional strings transform as they move. For this we must categorify the notion of a group! A '2-group' has objects:



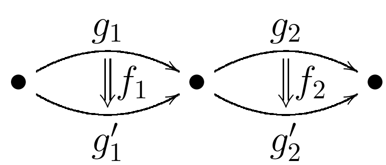
and also morphisms:



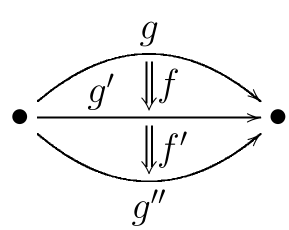
We can multiply objects:



multiply morphisms:



and also compose morphisms:



Various laws should hold... all dictated by the topology.

We can make this precise and categorify all of gauge theory. Today we'll do this for *trivial* bundles and 2-bundles; tomorrow for nontrivial ones.

Smooth Spaces

Alas, the category of smooth manifolds is a bit delicate:

- Given smooth manifolds X, Y, the space of smooth maps $f\colon X \to Y$ between is usually not a smooth manifold.
- Given smooth maps $f, g\colon X \to Y$, the solution set $\{f(x) = g(x)\} \subseteq X$ is usually not a smooth manifold.

So, let's use a more robust category! There are many choices. Just to be specific, let's use Chen's:

Let a **convex set** be a convex subset of $\mathbb{R}^n$ for any n.

Define a **smooth space** to be a set X with, for each convex set C, a collection of functions $\phi\colon C \to X$ called **plots** such that:

1. If $\phi\colon C \to X$ is a plot and $f\colon C' \to C$ is a smooth map between convex sets, then $\phi \circ f\colon C' \to X$ is a plot.
2. If $i_\alpha\colon C_\alpha \to C$ is an open cover of a convex set C by convex subsets C_α, and $\phi\colon C \to X$ has the property that $\phi \circ i_\alpha$ is a plot for all α, then ϕ is a plot.
3. Every map from a point to X is a plot.

Given smooth spaces X, Y, define a map $f\colon X \to Y$ to be **smooth** if $\phi \circ f\colon C \to Y$ is a plot whenever $\phi\colon C \to X$ is a plot.

Let C^∞ be the category of smooth spaces and smooth spaces. Then:

- C^∞ has limits and colimits, and the forgetful functor $C^\infty \to \mathrm{Set}$ preserves these. So, it has products $X \times Y$ and equalizers

$$\{f(x) = g(x)\} \subseteq X.$$

- C^∞ is cartesian closed. So, the space $C^\infty(X, Y)$ of smooth maps from X to Y is again smooth space, and

$$C^\infty(X \times Y, Z) \cong C^\infty(X, C^\infty(Y, Z)).$$

- Every finite-dimensional smooth manifold (possibly with boundary) is a smooth space; smooth maps between these are precisely those that are smooth in the usual sense.

- Every smooth space can be given the strongest topology in which all plots are continuous; smooth maps are then automatically continuous.

- Every subset of a smooth space is a smooth space.

- We can form a quotient of a smooth space X by any equivalence relation, and the result is again a smooth space.

- We can define vector fields and differential forms on smooth spaces, with many of the usual properties.

- Every simplicial set gives a smooth space whose de Rham cohomology matches its ordinary cohomology with $\mathbb{R}$ coefficients.

A nice category like this lets us develop *smooth homotopy theory!*

The Holonomy Along a Path

Let M be a smooth space. Let G be a **smooth group**: a smooth space that is a group with all the group operations being smooth (e.g. a Lie group). Let $\mathfrak{g}$ be the Lie algebra of G.

We want to compute a **holonomy** $\mathrm{hol}(\gamma) \in G$ for any path $\gamma \colon [t_0, t_1] \to M$. We seek to do this using a $\mathfrak{g}$-valued 1-form A on M, as follows:

Solve this differential equation:

$$\frac{d}{dt} g(t) = A(\gamma'(t))\, g(t)$$

with initial value $g(t_0) = 1$. Then let:

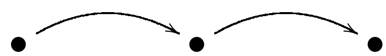
$$\mathrm{hol}(\gamma) = g(t_1).$$

We say the smooth group G is **exponentiable** if the above differential equation always has a smooth solution. For example: any Lie group is exponentiable, or any loop group $C^\infty(S^1, G)$ of a Lie group G.

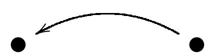
Henceforth, we assume all our smooth groups are exponentiable.

Holonomy as a Functor

The holonomy along a path doesn't depend on its parametrization. When we compose paths, their holonomies multiply:

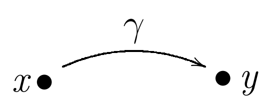


When we reverse a path, we get a path with the inverse holonomy:



So, let $\mathcal{P}_1(M)$ be the **path groupoid** of M:

- objects are points $x \in M$: $\bullet\, x$
- morphisms are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant near $t = 0, 1$:



This is a **smooth groupoid**: it has a smooth space of objects, a smooth space of morphisms, and all the groupoid operations are smooth.

Theorem. There is a one-to-one correspondence between smooth functors

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to G$$

and $\mathfrak{g}$-valued 1-forms A on M.

Internalization

Now let's categorify everything in sight and get a theory of holonomies for paths *and surfaces!*

The crucial trick is 'internalization', developed by Ehresmann in the 1960s. Given a familiar gadget x and a category K, we define an 'x in K' by writing the definition of x using commutative diagrams and interpreting these in K.

We need examples where $K = C^\infty$ is the category of smooth spaces:

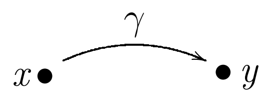
- A **smooth group** is a group in C^∞.
- A **smooth groupoid** is a groupoid in C^∞.
- A **smooth category** is a category in C^∞.
- A **smooth 2-group** is a 2-group in C^∞.
- A **smooth 2-groupoid** is a 2-groupoid in C^∞.
- A **smooth 2-category** is a 2-category in C^∞.

A category with all morphisms invertible is a groupoid. A groupoid with one object is a group. A 2-category with all morphisms and 2-morphisms invertible is a **2-groupoid**. A 2-groupoid with one object is a **2-group**.

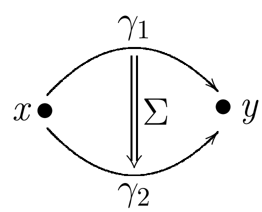
Here we only consider 'strict' 2-categories, hence strict 2-groupoids and 2-groups. Recall the definition....

A **2-category** has a set of objects:

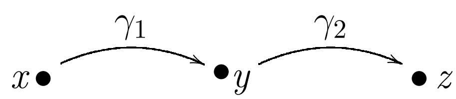
a set of morphisms:



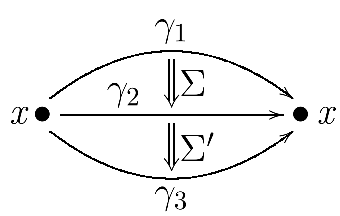
and a set of 2-morphisms:

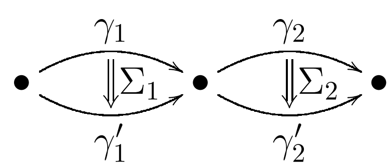


We can compose morphisms:

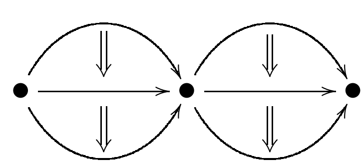


and compose 2-morphisms vertically and horizontally:





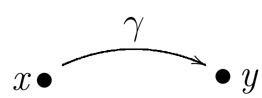
Each composition satisfies the unit law and associativity; they also obey the **interchange law**, which says this diagram gives a well-defined 2-morphism:



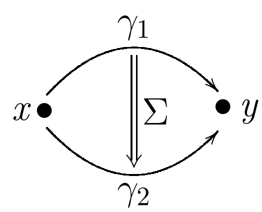
The Path 2-Groupoid

Just as holonomies along paths involve the path groupoid, holonomies over surfaces involve the **path 2-groupoid** $\mathcal{P}_2(M)$ of a smooth space M:

- objects are points of M: $\quad \bullet\, x$
- morphisms are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant in a neighborhood of $t = 0$ and $t = 1$:



- 2-morphisms are thin homotopy classes of smooth maps $\Sigma\colon [0,1]^2 \to M$ such that $\Sigma(s,t)$ is independent of s in a neighborhood of $s = 0$ and $s = 1$, and constant in a neighborhood of $t = 0$ and $t = 1$:



Theorem. For any smooth space M, $\mathcal{P}_2(M)$ is a smooth 2-groupoid.

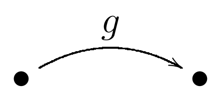
2-Groups

In higher gauge theory, holonomies takes values in a smooth 2-group!

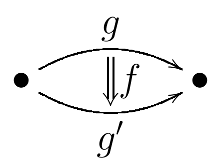
A 2-group $\mathcal{G}$ is a 2-groupoid with just one object:

$$\bullet$$

To reduce complexity, we can think of $\mathcal{G}$ as a category with objects like this:



and morphisms like this:



A 2-group is then the same as a strict monoidal category $(\mathcal{G}, \otimes, 1)$ where every morphism has an inverse, and also every object g has an inverse:

$$g \otimes g^{-1} = g^{-1} \otimes g = 1.$$

For example: any category C has an **automorphism 2-group** $\mathrm{AUT}(C)$, whose objects are invertible functors $g\colon C \to C$ and whose morphisms are natural isomorphisms $f\colon g \Rightarrow g'$ between these. We used this already in Schreier theory, in the case where C was a mere group.

Similarly, any smooth category C has a smooth 2-group $\mathrm{AUT}(C)$.

Crossed Modules

Any 2-group $\mathcal{G}$ is determined by:

- the group G consisting of all objects of $\mathcal{G}$,
- the group H consisting of all morphisms of $\mathcal{G}$ with source 1,
- the homomorphism $t\colon H \to G$ sending each morphism in H to its target,
- the action α of G on H defined using conjugation in the group of all morphisms of $\mathcal{G}$:

$$\alpha(g)h = 1_g h {1_g}^{-1}$$

The system (G, H, t, α) satisfies two equations making it into a **crossed module**:

$$t(\alpha(g)\,h) = g\,t(h)\,g^{-1} \quad \textbf{equivariance}$$

$$\alpha(t(h))\,h' = hh'h^{-1} \quad \textbf{the Peiffer identity}.$$

Conversely, crossed module gives a 2-group.

We can internalize this result: *smooth 2-groups are the same as smooth crossed modules!*

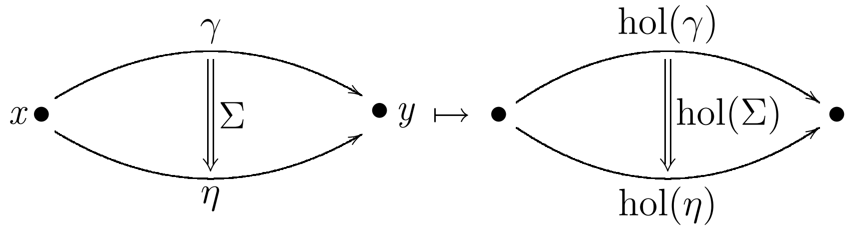
Differentiating everything in a smooth crossed module, we get a **differential crossed module** $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$.

Holonomy as a 2-Functor

Let M be a smooth space. Let $\mathcal{G}$ be a smooth 2-group, (G, H, t, α) its smooth crossed module and $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ its differential crossed module. Assume G and H are exponentiable.

Theorem. There is a one-to-one correspondence between smooth 2-functors

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$$



and pairs (A, B) consisting of a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B on M with vanishing **fake curvature**:

$$dA + A \wedge A + dt(B) = 0.$$

Punchline. When $\mathcal{G} = \mathrm{AUT}(H)$ for some Lie group H, the pair (A, B) is what Breen and Messing call a *connection on a trivial nonabelian H-gerbe.* The only difference is that they don't demand vanishing fake curvature. But, they don't get holonomies for surfaces!