Levin-Wen Models and Tensor Categories

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a joint work with Alexei Kitaev
Goals:

- to present a theory of boundary and defects of codimension 1,2,3 in non-chiral topological orders via Levin-Wen models;

- to show how the representation theory of tensor category enters the study of topological order at its full strength;

- to provide the physical foundation of the so-called extended Turaev-Viro topological field theories;
Kitaev’s Toric Code Model

Levin-Wen models

Extended Topological Field Theories
Outline

Kitaev’s Toric Code Model

Levin-Wen models

Extended Topological Field Theories
Kitaev’s Toric Code Model

- Kitaev’s Toric Code Model is equivalent to Levin-Wen model associated to the category $\text{Rep}_{\mathbb{Z}_2}$ of representations of $\mathbb{Z}_2$.

- It is the simplest example that can illustrate the general features of Levin-Wen models.
Kitaev’s Toric Code Model

\[ H = \bigotimes_{e \in \text{all edges}} \mathcal{H}_e; \quad \mathcal{H}_e = \mathbb{C}^2. \]

\[ H = - \sum_v A_v - \sum_p B_p. \]

\[ A_v = \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^4; \quad B_p = \sigma_z^5 \sigma_z^6 \sigma_z^7 \sigma_z^8. \]
Vacuum properties of toric code model:

A vacuum state $|0\rangle$ is a state satisfying $A_v |0\rangle = |0\rangle$, $B_p |0\rangle = |0\rangle$ for all $v$ and $p$.

- If surface topology is trivial (a sphere, an infinite plane), the vacuum is unique.

- Vacuum is given by the condensation of closed strings, i.e.

$$|0\rangle = \sum_{c \in \text{all closed string configurations}} |c\rangle.$$
The “set” of excitations determines the topological phase.

An excitation is defined to be super-selection sectors (irreducible modules) of a local operator algebra.

There are four types of excitations: 1, e, m, ϵ. We denote the ground states of these sectors as |0⟩, |e⟩, |m⟩, |ϵ⟩. We have

\[ \exists v_0, \quad A_{v_0} |e⟩ = -|e⟩, \]
\[ \exists p_0, \quad B_{p_0} |m⟩ = -|m⟩, \]
\[ \exists v_1, p_1, \quad A_{v_1} |ϵ⟩ = -|ϵ⟩, \quad B_{p_1} |ϵ⟩ = -|ϵ⟩. \]
$1 = e \otimes e \sim \sigma_z^1 \sigma_z^2 \sigma_z^3 \sigma_z^4 \sigma_z^5 |0\rangle,$

$1 = m \otimes m \sim \sigma_x^6 \sigma_x^7 \sigma_x^8 |0\rangle,$

$e \otimes m = \epsilon.$

$1, e, m, \epsilon$ are simple objects of a braided tensor category $Z(\text{Rep}_{\mathbb{Z}_2})$ which is the monoidal center of $\text{Rep}_{\mathbb{Z}_2}$.
This assignment actually gives a monoidal functor \( Z(\text{Rep}_{\mathbb{Z}_2}) \to \text{Rep}_{\mathbb{Z}_2} = \text{Fun}_{\text{Rep}_{\mathbb{Z}_2}}(\text{Rep}_{\mathbb{Z}_2}, \text{Rep}_{\mathbb{Z}_2}) \).
This assignment gives another monoidal functor $\mathcal{Z}(\text{Rep}_{\mathbb{Z}_2}) \to \text{Rep}_{\mathbb{Z}_2} = \text{Fun}_{\text{Rep}_{\mathbb{Z}_2}}(\text{Hilb}, \text{Hilb})$. 
defects of codimension 1, 2

\[ B_{p_1} = \sigma_x^7 \sigma_x^3 \sigma_x^2 \sigma_x^5; \quad B_{p_2} = \sigma_x^3 \sigma_x^7 \sigma_x^8 \sigma_x^9; \]

\[ B_Q = \sigma_x^6 \sigma_y^{17} \sigma_z^{18} \sigma_z^{19} \sigma_z^{20}. \]
defects of codimension 1

\[ 1 \mapsto 1 \mapsto 1, \quad e \xrightarrow{\sigma^3_z} \text{Ext}^{\text{defect}}_{3|7,8,9} \xrightarrow{\sigma^8_x} m, \]

\[ m \mapsto \text{Ext}^{\text{defect}}_{7|3,2,5} \mapsto e, \quad \epsilon \xrightarrow{\text{defect}} \text{Ext}^{\text{defect}}_{2,5,7,8,9,3} \mapsto \epsilon. \]

This assignment gives an invertible monoidal functor

\[ Z(\text{Rep}_{\mathbb{Z}_2}) \to \text{Fun}_{\text{Rep}_{\mathbb{Z}_2}|\text{Rep}_{\mathbb{Z}_2}}(\text{Hilb}, \text{Hilb}) \to Z(\text{Rep}_{\mathbb{Z}_2}). \]
Two eigenstates of $B_Q$ correspond to two simple $\text{Rep}_{\mathbb{Z}_2}$-$\text{Rep}_{\mathbb{Z}_2}$-bimodule functors $\text{Hilb} \rightarrow \text{Rep}_{\mathbb{Z}_2}$.
Outline

Kitaev’s Toric Code Model

Levin-Wen models

Extended Topological Field Theories
unitary tensor category $\mathcal{C} =$ unitary spherical fusion category

- semisimple: every object is a direct sum of simple objects;
- finite: there are only finite number of inequivalent simple objects, $i, j, k, l \in \mathcal{I}, |\mathcal{I}| < \infty$; $\dim \text{Hom}(A, B) < \infty$.
- monoidal: $(i \otimes j) \otimes k \cong i \otimes (j \otimes k);$ $1 \in \mathcal{I}, 1 \otimes i \cong i \cong i \otimes 1$;
- the fusion rule: $\dim \text{Hom}(i \otimes j, k) = N_{ij}^k$ is finite;
- $\mathcal{C}$ is not assumed to be braided.

**Theorem** (Müger): The monoidal center $Z(\mathcal{C})$ of $\mathcal{C}$ is a modular tensor category.
Fusion matrices

The associator \((i \otimes j) \otimes k \xrightarrow{\alpha} i \otimes (j \otimes k)\) induces an isomorphism:

\[
\text{Hom}((i \otimes j) \otimes k, l) \xrightarrow{\cong} \text{Hom}(i \otimes (j \otimes k), l)
\]

Writing in basis, we obtain the fusion matrices:

\[
F_{ijk; l} = \sum_n F_{mn}^{ijk; l}
\]  

(1)
Levin-Wen models

We fix a unitary tensor category $\mathcal{C}$ with simple objects $i, j, k, l, m, n \in \mathcal{I}$.

$\mathcal{H}_s = \mathcal{C}^{\mathcal{I}}$, $\mathcal{H}_v = \oplus_{i,j,k} \text{Hom}_\mathcal{C}(i \otimes j, k)$.

$\mathcal{H} = \otimes_s \mathcal{H}_s \otimes_v \mathcal{H}_v$. 

Figure: Levin-Wen model defined on a honeycomb lattice.
Hamiltonian

Chose a basis of $\mathcal{H}$, $i, j, k \in \mathcal{I}$ and $\alpha^{i', j'; k'} \in \text{Hom}_C(i' \otimes j', k')$, $\delta_{i, i'} \delta_{j, j'} \delta_{k, k'} \left( i, j; k | \alpha^{i', j'; k'} \right) = A_v |(i, j; k | \alpha^{i', j'; k'} \rangle)

If the spin on $v$ is such that $A_v$ acts as 1, then it is called stable.
The definition of $B_p$ operator

$$B_p := \sum_{i \in I} \frac{d_i}{\sum_k d_k^2} B_p^i$$

▶ If there are unstable spins around the plaquette $p$, $B_p^i$ act on the plaquette as zero.

▶ If all spins around the plaquette $p$ is stable, $B_p^i$ acts by inserting a loop labeled by $s \in I$ then evaluating the graph according to the composition of morphisms in $C$.

▶ $B_p$ is a projector. $A_v$ and $B_p$ commute.
Remark:

- Given a unitary tensor category $C$, we obtain a lattice model.

- Conversely, Levin-Wen showed how the axioms of the unitary tensor category can be derived from the requirement to have a fix-point wave function of a string-net condensation state.
Edge theories

If we cut the lattice, we automatically obtain a lattice with a boundary with all boundary strings labeled by simple objects in $\mathcal{C}$.

We will call such boundary as a $\mathcal{C}$-boundary or $\mathcal{C}$-edge.

**Question:** Are there any other possibilities?
It is possible to label the boundary strings by a different finite set \( \{\lambda, \sigma, \ldots\} \) which can be viewed as the set of inequivalent simples objects of another finite unitary semisimple category \( \mathcal{M} \).

The requirement of giving a fix-point wave function of string-net condensation state is equivalent to require that \( \mathcal{M} \) has a structure of \( C \)-module. We call such boundary an \( c\mathcal{M} \)-boundary or \( c\mathcal{M} \)-edge.
**C-module \( \mathcal{M} \):**

For \( i \in \mathcal{C}, \gamma, \lambda \in \mathcal{M} \),

- \( i \otimes \gamma \) is an object in \( \mathcal{M} \) (\( \otimes : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M} \))
- \( \dim \text{Hom}_\mathcal{M}(i \otimes \gamma, \lambda) = N_{i,\gamma}^\lambda < \infty \);
- \( 1 \otimes \gamma \cong \gamma \);
- associator \( (i \otimes j) \otimes \lambda \xrightarrow{\alpha} i \otimes (j \otimes \lambda) \);
- fusion matrices:

\[
\begin{align*}
\sum_n F_{ijk;\ell}^{mn} &= \sum_n \sum_{\sigma} F_{ijk;\ell}^{mn} \\
&= \sum_{\sigma} \sum_n F_{ijk;\ell}^{mn}
\end{align*}
\]
Excitations on boundary:

Two approaches:

1. Kitaev: excitations are super-selection sectors of a local operator algebra;

2. Levin-Wen: excitations can be classified by closed string operator which commute with the Hamiltonian.

゜ Above two approaches lead to the same results.
Levin-Wen approach

Close the boundary to a circle, a closed string operator on it is nothing but a systematic reassignment of boundary string labels and spin labels:

$$\gamma \mapsto F(\gamma) \in \mathcal{M},$$

$$\text{Hom}_\mathcal{M}(i \otimes \gamma, \lambda) \mapsto \text{Hom}_\mathcal{M}(i \otimes F(\gamma), F(\lambda))$$

This assignment is essentially the same data forming a functor from $\mathcal{M}$ to $\mathcal{M}$. Physical requirements (Levin-Wen) add certain consistency conditions which turn it into a $\mathcal{C}$-module functor.

**Theorem:** Excitations on a $\mathcal{C}\mathcal{M}$-edge are given by simple objects in the category $\text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{M})$ of $\mathcal{C}$-module functors.
Kitaev’s approach

We need construct the local operator algebra $A$.

$$A := \bigoplus_{i, \lambda_1, \lambda_2, \gamma_1, \gamma_2} \operatorname{Hom}_M(i \otimes \lambda_2, \lambda_1) \otimes \operatorname{Hom}_M(\gamma_1, i \otimes \gamma_2).$$

For $\xi \in \operatorname{Hom}_M(i \otimes \lambda_2, \lambda_1)$ and $\zeta \in \operatorname{Hom}_M(\gamma_1, i \otimes \gamma_2)$, the element $\xi \otimes \zeta \in A$ can be expressed by the following graph:

for $i \in C$ and $\lambda_1, \lambda_2, \gamma_1, \gamma_2 \in M$. 

[Diagram of the graph]
The multiplication $A \otimes A \rightarrow A$ is defined by

\[
\begin{align*}
\lambda_1 & \quad \xi & \quad \lambda_2 \\
\gamma_1 & \quad \xi & \quad \gamma_2 \\
\end{align*}
\begin{array}{c}
\uparrow i \\
\end{array}
\]

\[
\begin{align*}
\lambda'_1 & \quad \xi' & \quad \lambda'_2 \\
\gamma'_1 & \quad \xi' & \quad \gamma'_2 \\
\end{align*}
\begin{array}{c}
\uparrow j \\
\end{array}
\]

\[
\delta_{\lambda_2 \lambda'_2} \delta_{\gamma_2 \gamma'_2}
\]

where the last graph is a linear span of graphs in $A$ by applying F-moves twice and removing bubbles.
Action of $A$ on excitations

Figure: This picture shows how two elements of local operator algebra $A$ act on an edge excitation (up to an ambiguity of the excited region).
$A$ is bialgebra with above comultiplication. With some small modification, one can turn it into a weak $C^*$-Hopf algebra so that the boundary excitations form a finite unitary fusion category.
a defect line or a domain wall

\[ i, j, k, l \in \mathcal{C}, \lambda_1, \ldots, \lambda_9 \in \mathcal{M}, i', j', k', l' \in \mathcal{D}. \]  \[ \mathcal{C} \text{ and } \mathcal{D} \text{ are unitary tensor categories and } \mathcal{M} \text{ is a } \mathcal{C}-\mathcal{D}-\text{bimodule.} \]  \[ \text{We call such defect } \mathcal{C}\mathcal{M}\mathcal{D}\text{-defect line or } \mathcal{C}\mathcal{M}\mathcal{D}\text{-wall.} \]
A $\mathcal{M}$-edge can be viewed as $\mathcal{CM}_{\text{Hilb}}$-wall.

Conversely, if we fold the system along the $\mathcal{CM}_D$-wall, we obtain a doubled bulk system determined by $\mathcal{C} \boxtimes D^{\text{op}}$ with a single boundary determined by $\mathcal{M}$ which is viewed as a $\mathcal{C} \boxtimes D^{\text{op}}$-module.

$$\text{a } \mathcal{CM}_D\text{-wall} = \text{a } \mathcal{C} \boxtimes D^{\text{op}} \mathcal{M}\text{-edge}$$
Therefore, we have:

\[ \mathcal{C}\mathcal{M}_D \text{-wall excitations} = \mathcal{C}\otimes\mathcal{D}^{\text{op}}\mathcal{M} \text{-edge excitations} \]
\[ = \text{Fun}_{\mathcal{C}\otimes\mathcal{D}^{\text{op}}} (\mathcal{M}, \mathcal{M}) \]
\[ = \text{Fun}_{\mathcal{C}|\mathcal{D}} (\mathcal{M}, \mathcal{M}) \]

the category of \( \mathcal{C}\mathcal{D} \)-bimodule.

As a special case, a line in \( \mathcal{C} \)-bulk = a \( \mathcal{C}\mathcal{C} \)-wall.

\[ \mathcal{C} \text{-bulk excitations} = \mathcal{C}\mathcal{C} \text{-wall excitations} \]
\[ = \text{Fun}_{\mathcal{C}|\mathcal{C}} (\mathcal{C}, \mathcal{C}) = Z(\mathcal{C}) \]
A $cM_D$-wall can fuse with a $DN_E$-wall into a $c(M \boxtimes_D N)_E$-wall.

$cM_D$-wall (or $DN_E$-wall) excitations can fuse into $c(M \boxtimes_D N)_E$-wall as follow:

$$
(M \xrightarrow{F} M) \mapsto (M \boxtimes_D N \xrightarrow{F \boxtimes_D id_M} M \boxtimes_D N)
$$

$$
(N \xrightarrow{G} N) \mapsto (M \boxtimes_D N \xrightarrow{id_M \boxtimes_D G} M \boxtimes_D N)
$$
This assignment actually gives a monoidal functor $Z(\text{Rep}_{Z_2}) \to \text{Rep}_{Z_2} = \text{Fun}_{\text{Rep}_{Z_2}}(\text{Rep}_{Z_2}, \text{Rep}_{Z_2})$. 

\[
\begin{align*}
1 & \longrightarrow 1 \\
\epsilon & \longrightarrow e \\
m & \longrightarrow 1 \\
\epsilon & \longrightarrow e
\end{align*}
\]
This assignment gives another monoidal functor $Z(\text{Rep}_{\mathbb{Z}_2}) \to \text{Rep}_{\mathbb{Z}_2} = \text{Fun}_{\text{Rep}_{\mathbb{Z}_2}}(\text{Hilb}, \text{Hilb})$. 
This assignment gives an invertible monoidal functor $Z(\text{Rep}_{\mathbb{Z}_2}) \to \text{Fun}_{\text{Rep}_{\mathbb{Z}_2}|\text{Rep}_{\mathbb{Z}_2}}(\text{Hilb}, \text{Hilb}) \to Z(\text{Rep}_{\mathbb{Z}_2})$. 
Definition: If $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{C}$ and $\mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M} \cong \mathcal{D}$, then $\mathcal{M}$ and $\mathcal{N}$ are called invertible; $\mathcal{C}$ and $\mathcal{D}$ are called Morita equivalent.

- $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent iff $Z(\mathcal{C})$ is equivalent to $Z(\mathcal{D})$ as braided tensor categories.
- Invertible $\mathcal{C}$-$\mathcal{C}$-defects form a group called Picard group $\text{Pic}(\mathcal{C})$.  
- We denote the auto-equivalence of $Z(\mathcal{C})$ as $\text{Aut}(Z(\mathcal{C}))$.

Theorem (Kitaev-K., Etingof-Nikshych-Ostrik):

$$\text{Aut}(Z(\mathcal{C})) \cong \text{Pic}(\mathcal{C}).$$
Defects of codimension 2

- A defect of codimension 2 is a junction between two defect lines. It is given by a module functor.

- An excitation can be viewed as a defect of codimension 2.

- Conversely, a defect of codimension 2 is an excitation in the sense that it can be realized as a super-selection sector of a local operator algebra $A'$. 
Action of $A'$ on defects of codimension 2

$\lambda_1, \lambda_2, \lambda_3 \in \mathcal{M}, \gamma_1, \gamma_2, \gamma_3 \in \mathcal{N}$
Defects of codimension 3 (instantons)

If one takes into account the time direction, one can define a defect of codimension 3 by a natural transformation $\phi$ between module functors.

The Hamiltonian:

$$H \rightarrow H + H_t.$$  

where $H_t$ is a local operator defined using $\phi$. 
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<th>Ingredients in LW-model</th>
<th>Tensor-categorical notions</th>
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<td>a bulk lattice</td>
<td>a unitary tensor category $\mathcal{C}$</td>
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<td>string labels in a bulk</td>
<td>simple objects in a unitary tensor category $\mathcal{C}$</td>
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<tr>
<td>excitations in a bulk</td>
<td>simple objects in $Z(\mathcal{C})$ the monoidal center of $\mathcal{C}$</td>
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<tr>
<td>an edge</td>
<td>a $\mathcal{C}$-module $\mathcal{M}$</td>
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<td>string labels on an edge</td>
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<td>excitations on a $\mathcal{M}$-edge</td>
<td>$\text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{M})$: the category of $\mathcal{C}$-module functors</td>
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<td>bulk-excitations fuse into an $\mathcal{M}$-edge</td>
<td>$Z(\mathcal{C}) = \text{Fun}_{\mathcal{C}</td>
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<td></td>
<td>$(\mathcal{C} \xrightarrow{F} \mathcal{C}) \mapsto (\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M} \xrightarrow{F \boxtimes \text{id}<em>\mathcal{M}} \mathcal{C} \boxtimes</em>{\mathcal{C}} \mathcal{M})$.</td>
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## Dictionary 2:

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<tr>
<td>a domain wall</td>
<td>a $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{N}$</td>
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<tr>
<td>string labels on a $\mathcal{N}$-wall</td>
<td>simple objects in a $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{C}\mathcal{N}_D$</td>
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<tr>
<td>excitations on a $\mathcal{N}$-wall</td>
<td>$\text{Fun}_{\mathcal{C}</td>
</tr>
<tr>
<td>fusion of two walls</td>
<td>$\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$</td>
</tr>
<tr>
<td>an invertible $\mathcal{C}\mathcal{N}_D$-wall</td>
<td>$\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent, i.e. $\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{N}^{\text{op}} \cong \mathcal{C}$, $\mathcal{N}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{D}$.</td>
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<tr>
<td>defects of codimension 2: a $\mathcal{M}$-$\mathcal{N}$-excitation</td>
<td>simple objects $\mathcal{F}, \mathcal{G} \in \text{Fun}_{\mathcal{C}</td>
</tr>
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<td>a defect of codimension 3 or an instanton</td>
<td>a natural transformation $\phi : \mathcal{F} \rightarrow \mathcal{G}$</td>
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Outline

Kitaev’s Toric Code Model

Levin-Wen models

Extended Topological Field Theories
Extended topological field theories was formulated by Baez and Dolan in terms of n-category in 90s. The classification was given in the so-called Baez-Dolan conjecture which was recently proved by Lurie.

Levin-Wen models enriched by defects of codimension 1,2,3 provides a physical foundation behind the so-called extended Turaev-Viro topological field theories.
The building blocks of the lattice models:

\[\begin{array}{c}
\text{F} \downarrow \downarrow \text{G} \\
\phi \rightarrow \rightarrow \phi' \\
\text{C} \downarrow \downarrow \text{D} \\
\text{M} \\
\phi \\
\text{N} \\
\end{array}\]

which 0-1-2-3 cells of a tri-category, or “equivalently”,

\[\begin{array}{c}
\text{C-Mod} \phi \rightarrow \rightarrow \phi' \text{D-Mod} \\
\text{C-Mod} \phi \rightarrow \rightarrow \phi' \text{D-Mod} \\
\end{array}\]
Excitations (topological phases):

\[ Z(\mathcal{M}) \]

\[ Z(\mathcal{C}) \quad Z(\mathcal{M}, \mathcal{N})_\mathcal{F} \quad Z(\phi) \quad Z(\mathcal{M}, \mathcal{N})_\mathcal{G} \quad Z(\mathcal{D}) \]

\[ Z(\mathcal{N}) \]

\[ Z(\mathcal{M}) := \text{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{M}), \quad Z(\mathcal{N}) := \text{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{N}, \mathcal{N}), \]

\[ \mathcal{F}, \mathcal{G}, \in \text{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{N}), \quad Z(\mathcal{M}, \mathcal{N})_\mathcal{F} := Z(\mathcal{N}) \circ \mathcal{F} \circ Z(\mathcal{M}), \]

\[ Z(\mathcal{M}, \mathcal{N})_\mathcal{G} := Z(\mathcal{N}) \circ \mathcal{G} \circ Z(\mathcal{M}). \]
**Conjecture** (Functoriality of Holography): The assignment $Z$ is a functor between two tricategories.

**Remark:** It also says that the notion of monoidal center is functorial.
**General philosophy**: for $n+1$-dim extended TQFT,

$$\text{pt} \mapsto n\text{-category of boundary conditions}.$$ 

**Extended Turaev-Viro (2+1) TQFT**: the bicategory of boundary conditions of LW-models $= \mathcal{C}\text{-Mod},$

$$\text{pt}_{+, -} \mapsto \mathcal{C}, \mathcal{D} \text{ or } (\mathcal{C}\text{-Mod} \cong \mathcal{D}\text{-Mod}),$$

an interval $\mapsto \mathcal{M}_\mathcal{D}, \mathcal{N}_\mathcal{C} \text{ (invertible)}$

$$S^1 \mapsto Tr(\mathcal{C}) = Z(\mathcal{C}),$$

Conjecturely,

$$\text{Turaev-Viro}(\mathcal{C}) = \text{Reshtikin-Turaev}(Z(\mathcal{C})).$$
Thank you!