Higher Gauge Theory

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Abstract

Just as gauge theory describes the parallel transport of point particles using connections on bundles, higher gauge theory describes the parallel transport of 1-dimensional objects (e.g. strings) using 2-connections on 2-bundles. A 2-bundle is a categorified version of a bundle: that is, one where the fiber is not a manifold but a category with a suitable smooth structure. Where gauge theory uses Lie groups and Lie algebras, higher gauge theory uses their categorified analogues: Lie 2-groups and Lie 2-algebras. We describe a theory of 2-connections on principal 2-bundles and explain how this is related to Breen and Messing's theory of connections on nonabelian gerbes. The distinctive feature of our theory is that a 2-connection allows parallel transport along paths and surfaces in a parametrization-independent way. In terms of Breen and Messing's framework, this requires that the 'fake curvature' must vanish. In this paper we summarize the main results of our theory without proofs.

1 Introduction

Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign a group element
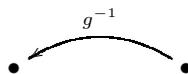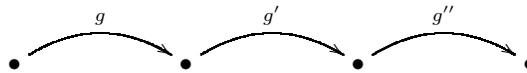
to each path:

$$g$$
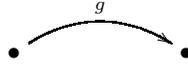
The reason is that composition of paths then corresponds to multiplication in the group:

$$g \qquad g'$$

while reversing the direction of a path corresponds to taking inverses:

$$g^{-1}$$

and the associative law makes the holonomy along a triple composite unambiguous:

$$g \qquad g' \qquad g''$$

In short, the topology dictates the algebra!

Now suppose we wish to do something similar for 1-dimensional 'strings' that trace out 2-dimensional surfaces as they move. Naively we might wish our holonomy to assign a group element to each surface like this:
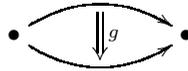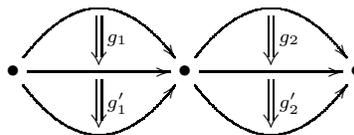
There are two obvious ways to compose surfaces of this sort, vertically:
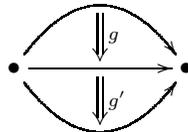
and horizontally:

Suppose that both of these correspond to multiplication in the group G. Then to obtain a well-defined holonomy for this surface regardless of whether we do vertical or horizontal composition first:

we must have
$$(g_1 g_1')(g_2 g_2') = (g_1 g_2)(g_1' g_2').$$
This forces G to be abelian!

In fact, this argument goes back to a classic paper by Eckmann and Hilton [1]. They showed that even if we allow G to be equipped with two products, say gg' for vertical composition and $g \circ g'$ for horizontal, so long as both products share the same unit and satisfy this 'interchange law':
$$(g_1 g_1') \circ (g_2 g_2') = (g_1 \circ g_2)(g_1' \circ g_2')$$
then in fact they must agree — so by the previous argument, both are abelian. The proof is very easy:
$$g \circ g' = (g1) \circ (1g') = (g \circ 1)(1 \circ g') = gg'$$

Pursuing this approach, we would ultimately reach the theory of connections on abelian gerbes [2-8]. If $G = U(1)$, such a connection can be locally identified with a 2-form — but globally it is a subtler object, just as a connection on a U(1) bundle can be locally identified with a 1-form, but not globally. In fact, connections on abelian gerbes play an important role in string theory [9-11]. Just as ordinary electromagnetism is described by a connection on a U(1) bundle, usually called the 'vector potential' and denoted A, the stringy analogue of electromagnetism is described by a connection on a U(1) gerbe, called the B field.

To go beyond this and develop a theory of *nonabelian* higher gauge fields, we must let the topology dictate the algebra. Readers familiar with higher categories will already have noticed that 1-dimensional pictures above resemble diagrams in category theory, while the 2-dimensional pictures resemble diagrams in 2-category theory. This suggests that the holonomies in higher gauge theory should take values in some 'categorified' analogue of a Lie group — that is, some gadget resembling a Lie group, but which is a category rather than a set. We call this 'Lie 2-group'.

In fact, Lie 2-groups and their Lie 2-algebras have already been studied [12, 13] and interesting examples have been constructed using the mathematics of string theory: central extensions of loop groups [14]. But even without knowing this, we could be led to the definition of a Lie 2-group by considering a kind of connection that gives holonomies *both for paths and for surfaces.*

So, let us assume that for each path we have a holonomy taking values in some Lie group G, where composition of paths corresponds to multiplication in G. Assume also that for each 1-parameter family of paths with fixed endpoints we have a holonomy taking values in some other Lie group H, where vertical composition corresponds to multiplication in H:
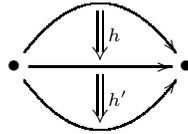
Next, assume that we can parallel transport an element $g \in G$ along a 1-parameter family of paths to get a new element $g' \in G$:
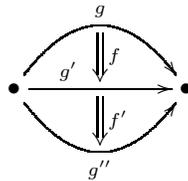
$$\bullet \quad \overset{\textstyle g}{\underset{\textstyle g'}{\Downarrow h}} \quad \bullet$$
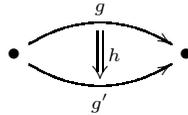
This picture suggests that we should think of h as a kind of 'arrow' or 'morphism' going from g to g'. We can use categories to formalize this. In category theory, when a morphism goes from an object x to an object y, we think of the morphism as determining both its source x and its target y. The group element h does not determine g or g'. However, the pair (g, h) does.

For this reason, it is useful to create a category $\mathcal{G}$ where the set of objects, say $\mathrm{Ob}(\mathcal{G})$, is just G, while the set of morphisms, say $\mathrm{Mor}(\mathcal{G})$, consists of ordered pairs $f = (g, h) \in G \times H$. Switching our notation to reflect this, we rewrite the above picture as
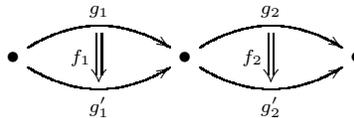
$$\bullet \quad \overset{\textstyle g}{\underset{\textstyle g'}{\Downarrow f}} \quad \bullet$$
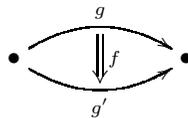
and write $f \colon g \to g'$ for short.

In this new notation, we can vertically compose $f \colon g \to g'$ and $f' \colon g' \to g''$ to get $f f' \colon g \to g''$, as follows:

$$\bullet \quad \overset{\textstyle g}{\underset{\textstyle g''}{\overset{\Downarrow f}{\underset{\Downarrow f'}{g'}}}} \quad \bullet$$

This is just composition of morphisms in the category $\mathcal{G}$. However, we can also horizontally compose $f_1 \colon g_1 \to g_1'$ and $f_2 \colon g_2 \to g_2'$ to get $f_1 \circ f_2 \colon g_1 g_2 \to g_1' g_2'$, as follows:

$$\bullet \quad \overset{\textstyle g_1}{\underset{\textstyle g_1'}{f_1 \Downarrow}} \quad \bullet \quad \overset{\textstyle g_2}{\underset{\textstyle g_2'}{f_2 \Downarrow}} \quad \bullet$$

We assume this operation makes $\mathrm{Mor}(\mathcal{G})$ into a group with the pair $(1, 1) \in G \times H$ as its multiplicative unit.

The good news is that now we can assume an interchange law saying this

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holonomy is well-defined:
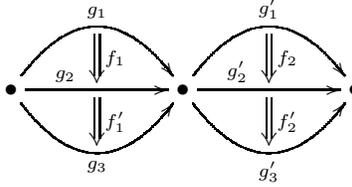
namely:

$$(f_1 f_1') \circ (f_2 f_2') = (f_1 \circ f_2)(f_1' \circ f_2')$$

without forcing either G or H to be abelian! Instead, the group $\mathrm{Mor}(\mathcal{G})$ is forced to be a semidirect product of G and H.

The structure we are rather roughly describing here turns out to be none other than a 'Lie 2-group'. This is an 'internal category' in the category of Lie groups. In other words, it is a category where the set of objects is a Lie group, the set of morphisms is a Lie group, and all the usual category operations are Lie group homomorphisms.

This audacious process — taking a familiar mathematical concept defined using sets and function and transplanting it to live within some other category — is far from new. For example, a Lie group is a group in Diff, the category of smooth manifolds and smooth maps. The general idea of an 'internal category' living within some other category was described by Charles Ehresmann [15] in the early 1960's. In the next section we begin by reviewing this idea. In the rest of the paper, we use it to categorify the theory of Lie groups, Lie algebras, bundles and connections. Before we proceed, let us sketch our overall plan.

The starting point is the ordinary concept of a principal fiber bundle. Such a bundle can be specified using the following 'gluing data':

- a base manifold M,

- a cover of M by open sets $\{U_i\}_{i \in I}$,

- a Lie group G (the 'gauge group' or 'structure group'),

- on each double overlap $U_{ij} = U_i \cap U_j$ a G-valued function g_{ij},

- such that on triple overlaps the following transition law holds:

$$g_{ij} g_{jk} = g_{ik}.$$

Such a bundle is augmented with a connection by specifying:

- in each open set U_i a smooth functor $\mathrm{hol}_i \colon \mathcal{P}_1(U_i) \to G$ from the path groupoid of U_i to the gauge group,

- such that for all paths γ in double overlaps U_{ij} the following transition law holds:

$$\mathrm{hol}_i(\gamma) \;=\; g_{ij} \, \mathrm{hol}_j(\gamma) \, g_{ij}^{-1}.$$

Here the 'path groupoid' $\mathcal{P}_1(M)$ of a manifold M has points of M as objects and certain equivalence classes of smooth paths in M as morphisms. There are various ways to work out the technical details and make $\mathcal{P}_1(M)$ into a 'smooth groupoid'; here we follow the approach of Barrett [16], who uses 'thin homotopy classes' of paths. Technical details aside, the basic idea is that a connection on a trivial G-bundle gives a well-behaved map assigning to each path γ in the base space the holonomy $\mathrm{hol}(\gamma) \in G$ of the connection along that path. Saying this map is a 'smooth functor' means that these holonomies compose when we compose paths, and that the holonomy $\mathrm{hol}(\gamma)$ depends smoothly on the path γ.

A basic goal of higher gauge theory is to categorify all of this and to work out the consequences. As mentioned, the key tool is internalization. This leads us immediately to the concept of a Lie 2-group, and also to that of a 'smooth 2-space': a category in Diff, or more generally in some category of smooth spaces that allows for infinite-dimensional examples.

Using these concepts, Bartels [17] has defined a 'principal 2-bundle E over M with structure 2-group $\mathcal{G}$'. To arrive at this definition, the key steps are to replace the total space E and base space M of a principal bundle by smooth 2-spaces, and to replace the structure group by a Lie 2-group. In this paper we only consider the case where M is an ordinary space, which can be regarded as a 2-space with only identity morphisms. We show that for a suitable choice of structure 2-group, principal 2-bundles give abelian gerbes over M. For another choice, they give nonabelian gerbes. This sets the stage for a result relating the 2-bundle approach to higher gauge theory to Breen and Messing's approach based on nonabelian gerbes [18].
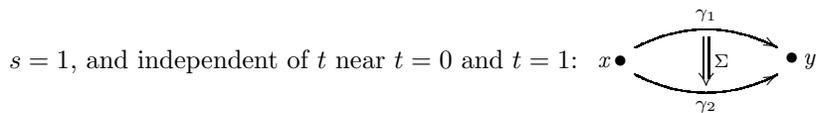
Just as a connection on a trivial principal bundle over M gives a functor from the path groupoid of M to the structure group, one might hope that a '2-connection' on a trivial principal 2-bundle would define a 2-functor from some sort of 'path 2-groupoid' to the structure 2-group. This has already been confirmed in the context of higher lattice gauge theory [19-21]. Thus, the main issues not yet addressed are those involving differentiability.

To address these issues, we define for any smooth space M a smooth 2-groupoid $\mathcal{P}_2(M)$ such that:

- the objects of $\mathcal{P}_2(M)$ are points of M: $\bullet\, x$

- the morphisms of $\mathcal{P}_2(M)$ are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(s)$ is constant in a neighborhood of $s = 0$

  and $s = 1$: $x \bullet \xrightarrow{\ \gamma\ } \bullet\, y$
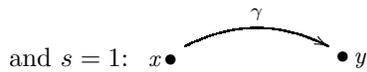
- the 2-morphisms of $\mathcal{P}_2(M)$ are 'bigons': that is, thin homotopy classes of smooth maps $\Sigma\colon [0,1]^2 \to M$ such that $\Sigma(s,t)$ is constant near $s = 0$ and

  $s = 1$, and independent of t near $t = 0$ and $t = 1$: $x \bullet \overset{\gamma_1}{\underset{\gamma_2}{\Longrightarrow \Sigma}} \bullet\, y$

The 'thin homotopy' equivalence relation, borrowed from the work of Mackaay and Picken [7, 22], guarantees that two maps differing only by a reparametrization define the same bigon. This is important because we seek a *reparametrization-invariant notion of surface holonomy*.

We define a '2-connection' on a trivial principal 2-bundle over M to be a smooth 2-functor $\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$, where $\mathcal{G}$ is the structure 2-group. This means that the 2-connection assigns holonomies both to paths and bigons, independent of their parametrization, compatible with the standard operations of composing paths and bigons, and depending smoothly on the path or surface in question.

We also define 2-connections for nontrivial principal 2-bundles, and state a theorem obtaining these from Lie-algebra-valued differential forms. We then show that for a certain class of structure 2-groups, such differential forms reduce to Breen and Messing's 'connections on nonabelian gerbes' [18]. The surprise is that we only obtain connections satisfying a certain constraint: the 'fake curvature' must vanish!

To understand this, one must recall [12] that a Lie 2-group $\mathcal{G}$ amounts to the same thing as a 'crossed module' of Lie groups (G, H, t, α), where:

- G is the group of objects of G, $\mathrm{Ob}(\mathcal{G})$:

- H is the subgroup of $\mathrm{Mor}(\mathcal{G})$ consisting of morphisms with source equal to $1 \in G$:

- $t\colon H \to G$ is the homomorphism sending each morphism in H to its target,

- α is the action of G as automorphisms of H defined using conjugation in $\mathrm{Mor}(\mathcal{G})$ as follows: $\alpha(g)h = 1_g h 1_g{}^{-1}$.

Differentiating all this data one obtains a 'differential crossed module' $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$, which is just another way of talking about a Lie 2-algebra [23].

In these terms, a 2-connection on a trivial principal 2-bundle over M with structure 2-group $\mathcal{G}$ consists of a $\mathfrak{g}$-valued 1-form A together with an $\mathfrak{h}$-valued 2-form B on M. Translated into this framework, Breen and Messing's 'fake curvature' is the $\mathfrak{g}$-valued 2-form

$$dt(B) + F_A,$$

where $F_A = \mathbf{d}A + A \wedge A$ is the usual curvature of A. We show that *if and only if the fake curvature vanishes*, one obtains a well-defined 2-connection $\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$.

The importance of vanishing fake curvature in the framework of lattice gauge theory was already emphasized by Girelli and Pfeiffer [21]. The special case where also $F_A = 0$ was studied by Alvarez, Ferreira, Sanchez and Guillen [24]. The case where $G = H$ has been studied already by the second author of this paper [25]. Our result subsumes these cases in a common framework.

This paper is an introduction to work in progress [23], which began in rudimentary form as an article by the first author [26], and overlaps to some extent with theses by Bartels [17] and the second author [27]. Bartels' thesis develops

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the general theory of 2-bundles. The second author's thesis investigates the relationship between nonabelian higher gauge theory and the physics of strings. Aschieri, Cantini and Jurčo [28, 29] have also studied this subject, using connections on nonabelian gerbes. Other physicists, including Chepelev [30] and Hofman [31], have also studied nonabelian higher gauge fields.

2 Internalization

The idea of internalization is simple: given a mathematical concept X defined solely in terms of sets, functions and commutative diagrams involving these, and given some category K, one obtains the concept of an 'X in K' by replacing all these sets, functions and commutative diagrams by corresponding objects, morphisms, and commutative diagrams in K.

The case we need here is when X is the concept of 'category':

Definition 1. *Let K be a category. An* **internal category in** K, *or simply* **category in** K, *say C, consists of:*

- *an object* $\mathrm{Ob}(C) \in K$,

- *an object* $\mathrm{Mor}(C) \in K$,

- **source** *and* **target** *morphisms* $s, t \colon \mathrm{Mor}(C) \to \mathrm{Ob}(C)$,

- *an* **identity-assigning** *morphism* $i \colon \mathrm{Ob}(C) \to \mathrm{Mor}(C)$,

- *a* **composition** *morphism* $\circ \colon \mathrm{Mor}(C)_s \times_t \mathrm{Mor}(C) \to \mathrm{Mor}(C)$

satisfying the usual rules of a category expressed in terms of commutative diagrams.

Here $\mathrm{Mor}(C)_s \times_t \mathrm{Mor}(C)$ is defined using a pullback: if K is the category of sets, it is the set of composable pairs of morphisms in C. Inherent in the definition is the assumption that this pullback exist, along with the other pullbacks needed to write the rules of a category as commutative diagrams.

We can similarly define a **functor in** K and a **natural transformation in** K; details can be found in Borceux's handbook [32]. There is a 2-category KCat whose objects, morphisms and 2-morphisms are categories, functors and natural transformations in K. To study *symmetries* in higher gauge theory, we need these examples:

Definition 2. *Let* LieGrp *be the category whose objects are Lie groups and whose morphisms are Lie group homomorphisms. Then the objects, morphisms and 2-morphisms of* LieGrpCat *are called* **Lie 2-groups**, **Lie 2-group homomorphisms**, *and* **Lie 2-group 2-homomorphisms**, *respectively.*

Definition 3. *Let* LieAlg *be the category whose objects are Lie algebras and whose morphisms are Lie algebra homomorphisms. Then the objects, morphisms and 2-morphisms of* LieAlgCat *are called* **Lie 2-algebras**, **Lie 2-algebra homomorphisms**, *and* **Lie 2-algebra 2-homomorphisms**, *respectively.*

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For the benefit of experts, we should admit that we are only defining 'strict' Lie 2-groups and Lie 2-algebras, where all the usual laws hold as equations. We rarely need any other kind in this paper, but there are more general Lie 2-groups and Lie 2-algebras where the usual laws hold only up to isomorphism [12, 13].

We could also consider categories in Diff, the category whose objects are finite-dimensional smooth manifolds and whose morphisms are smooth maps. Ehresmann [33] introduced these in the late 1950's under the name of 'differentiable categories'. However, these are not quite what we want here, for two reasons. First, unlike LieGrp and LieAlg, Diff does not have pullbacks in general. This means that when we try to define a category in Diff, the set of composable pairs of morphisms is not automatically a smooth manifold. Second, the space of smooth paths in a smooth manifold is not again a smooth manifold. This is an annoyance when studying connections on bundles.

To solve these problems, we want a category of 'smooth spaces' that has pullbacks and includes path spaces. Various categories of this sort have been proposed. It is unclear which one is best, but we shall use the last of several variants proposed by Chen [34, 35]. In what follows, we use **convex set** to mean a convex subset of $\mathbb{R}^n$, where n is arbitrary (not fixed). Any convex set inherits a topology from its inclusion in $\mathbb{R}^n$. We say a map f between convex sets is **smooth** if arbitrarily high derivatives of f exist and are continuous, using the usual definition of derivative as a limit of a quotient.

Definition 4. *A **smooth space** is a set X equipped with, for each convex set C, a collection of functions $\phi: C \to X$ called **plots** in X, such that:*

1. *If $\phi: C \to X$ is a plot in X, and $f: C' \to C$ is a smooth map between convex sets, then $\phi \circ f$ is a plot in X,*

2. *If $i_\alpha: C_\alpha \to C$ is an open cover of a convex set C by convex subsets C_α, and $\phi: C \to X$ has the property that $\phi \circ i_\alpha$ is a plot in X for all α, then ϕ is a plot in X.*

3. *Every map from a point to X is a plot in X.*

Definition 5. *A **smooth map** from the smooth space X to the smooth space Y is a map $f: X \to Y$ such that for every plot ϕ in X, $\phi \circ f$ is a plot in Y.*

In highbrow lingo, this says that smooth spaces are sheaves on the category whose objects are convex sets and whose morphisms are smooth maps, equipped with the Grothendieck topology where a cover is an open cover in the usual sense. However, smooth spaces are not arbitrary sheaves of this sort, but precisely those for which two plots with domain C agree whenever they agree when pulled back along every smooth map from a point to C.

Using this, it is straightforward to check that there is a category C^∞ whose objects are smooth spaces and whose morphisms are smooth maps. Moreover this category is cartesian closed, and it has arbitrary limits and colimits. It also has other nice properties:

- Every finite-dimensional smooth manifold (possibly with boundary) is a smooth space; smooth maps between these are precisely those that are smooth in the usual sense.

- Every smooth space can be given the strongest topology in which all plots are continuous; smooth maps are then automatically continuous.

- Every subset of a smooth space is a smooth space.

- We can form a quotient of a smooth space X by any equivalence relation, and the result is again a smooth space.

- We can define vector fields and differential forms on smooth spaces, with many of the usual properties.

With the notion of smooth space in hand, we can make the following definitions:

Definition 6. *Let* C^∞ *be the category whose objects are smooth spaces and whose morphisms are smooth maps. Then the objects, morphisms and 2-morphisms of* $\mathrm{C}^\infty\mathrm{Cat}$ *are called* **smooth 2-spaces**, **smooth maps**, *and* **smooth 2-maps**, *respectively.*

Writing down the above definitions is quick and easy. It takes longer to understand them and apply them to higher gauge theory. For this we must unpack them and look at examples.

To get examples of Lie 2-groups, we can use 'Lie crossed modules'. A **Lie crossed module** is a quadruple (G, H, t, α) where G and H are Lie groups, $t: H \to G$ is a Lie group homomorphism and α is a smooth action of G as automorphisms of H such that t is equivariant:

$$t(\alpha(g)(h)) = g\, t(h)\, g^{-1}$$

and satisfies the so-called 'Peiffer identity':

$$\alpha(t(h))(h') = hh'h^{-1}.$$

We obtain a Lie crossed module from a Lie 2-group $\mathcal{G}$ as follows:

- G is the Lie group of objects of G, $\mathrm{Ob}(\mathcal{G})$,

- H is the subgroup of $\mathrm{Mor}(\mathcal{G})$ consisting of morphisms with source equal to $1 \in G$:

- $t: H \to G$ is the homomorphism sending each morphism in H to its target,

- α is the action of G as automorphisms of H defined using conjugation in $\mathrm{Mor}(\mathcal{G})$ as follows: $\alpha(g)h = 1_g h 1_g{}^{-1}$.

Conversely, we can reconstruct any Lie 2-group from its Lie crossed module. In fact, the 2-category of Lie 2-groups is biequivalent to that of Lie crossed modules [12]. This gives various examples:

Example 7. Given any abelian group H, there is a Lie crossed module where G is the trivial group and t, α are trivial. This gives a Lie 2-group $\mathcal{G}$ with one object and H as the group of morphisms. Lie 2-groups of this sort are important in the theory of *abelian* gerbes.

Example 8. Given any Lie group H, there is a Lie crossed module with $G = \text{Aut}(H)$, $t: H \to G$ the homomorphism assigning to each element of H the corresponding inner automorphism, and the obvious action of G as automorphisms of H. We call the corresponding Lie 2-group the **automorphism 2-group** of H, and denote it by $\mathcal{AUT}(H)$. This sort of 2-group is important in the theory of *nonabelian* gerbes.

We use the term 'automorphism 2-group' because $\mathcal{AUT}(H)$ really is the 2-group of symmetries of H. Lie groups form a 2-category, any object in a 2-category has a 2-group of symmetries, and the 2-group of symmetries of H is naturally a Lie 2-group, which is none other than $\mathcal{AUT}(H)$. See [12] for details.

Example 9. Suppose that $1 \to A \hookrightarrow H \xrightarrow{t} G \to 1$ is a central extension of the Lie group G by the Lie group H. Then there is a Lie crossed module with this choice of $t: H \to G$. To construct α we pick any section s, that is, any function $s: G \to H$ with $t(s(g)) = g$, and define

$$\alpha(g)h = s(g)hs(g)^{-1}.$$

Since A lies in the center of H, α independent of the choice of s. We do not need a global smooth section s to show $\alpha(g)$ depends smoothly on g; it suffices that there exist a local smooth section in a neighborhood of each $g \in G$.

It is easy to generalize this idea to infinite-dimensional cases if we work not with Lie groups but **smooth groups**: that is, groups in the category of smooth spaces. The basic theory of smooth groups, smooth 2-groups and smooth crossed modules works just like the finite-dimensional case, but with the category of smooth spaces replacing Diff. In particular, every smooth group G has a Lie algebra $\mathfrak{g}$.

Given a connected and simply-connected compact simple Lie group G, the loop group ΩG is a smooth group. For each 'level' $k \in \mathbb{Z}$, this group has a central extension

$$1 \to \text{U}(1) \hookrightarrow \widehat{\Omega_k G} \xrightarrow{t} \Omega G \to 1$$

as explained by Pressley and Segal [36]. The above diagram lives in the category of smooth groups, and there exist local smooth sections for $t: \widehat{\Omega_k G} \to \Omega G$, so we obtain a smooth crossed module $(\Omega G, \widehat{\Omega_k G}, t, \alpha)$ with α given as above. This in turn gives an smooth 2-group which we call the **level-k loop 2-group** of G, $\mathcal{L}_k G$.

It has recently been shown [14] that $\mathcal{L}_k G$ fits into an exact sequence of smooth 2-groups:

$$1 \to \mathcal{L}_k G \hookrightarrow \mathcal{P}_k G \longrightarrow G \to 1$$

where the middle term, the **level-k path 2-group** of G, has very interesting properties. In particular, when $k = \pm 1$, the geometric realization of the nerve of

$\mathcal{P}_k G$ is a topological group that can also be obtained by killing the 3rd homotopy group of G. When $G = \mathrm{Spin}(n)$, this topological group goes by the name of $\mathrm{String}(n)$, since it plays a role in defining spinors on loop space [37]. The group $\mathrm{String}(n)$ also shows up in Stolz and Teichner's work on elliptic cohomology, which involves a notion of parallel transport over surfaces [38]. So, we expect that $\mathcal{P}_k G$ will be an especially interesting structure 2-group for applications of 2-bundles to string theory.

To define the holonomy of a connection, we need smooth groups with an extra property: namely, that for every smooth function $f\colon [0,1] \to \mathfrak{g}$ there is a unique smooth function $g\colon [0,1] \to G$ solving the differential equation

$$\frac{d}{dt} g(t) = f(t) g(t)$$

with $g(0) = 1$. We call such smooth groups **exponentiable**. Similarly, we call a smooth 2-group $\mathcal{G}$ **exponentiable** if its crossed module (G, H, t, α) has both G and H exponentiable. In particular, every Lie group and thus every Lie 2-group is exponentiable. The smooth groups ΩG and $\widehat{\Omega_k G}$ are also exponentiable, as are the 2-groups $\mathcal{L}_k G$ and $\mathcal{P}_k G$. So, for the convenience of stating theorems in a simple way, *we henceforth implicitly assume all smooth groups and 2-groups under discussion are exponentiable.* We only really need this in Theorems 21 and 23.

Finally, here are some easy examples of smooth 2-spaces:

Example 10. Any smooth space can be seen as a smooth 2-space with only identity morphisms.

Example 11. Any smooth group (for example a Lie group) can be seen as a smooth 2-space with only one object.

Example 12. Given a smooth space M, there is a smooth 2-space $\mathcal{P}_1(M)$, the **path groupoid of** M, such that:

- the objects of $\mathcal{P}_1(M)$ are points of M,

- the morphisms of $\mathcal{P}_1(M)$ are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(s)$ is constant near $s = 0$ and $s = 1$.

Here a **thin homotopy** between smooth paths $\gamma_1, \gamma_2\colon [0,1] \to M$ is a smooth map $H\colon [0,1]^2 \to M$ such that:

- $H(s,0) = \gamma_1(s)$ and $H(s,1) = \gamma_2(s)$,

- $H(s,t)$ is independent of t near $t = 0$ and near $t = 1$,

- $H(s,t)$ is constant near $s = 0$ and near $s = 1$,

- the rank of the differential $dH(s,t)$ is < 2 for all $s, t \in [0,1]$.

The last condition is what makes the homotopy 'thin': it guarantees that the homotopy sweeps out a surface of vanishing area.

To see how $\mathcal{P}_1(M)$ becomes a 2-space, first note that the space of smooth maps $\gamma\colon[0,1]\to M$ becomes a smooth space in a natural way, as does the subspace satisfying the constancy conditions near $t=0,1$, and finally the quotient of this subspace by the thin homotopy relation. This guarantees that $\mathrm{Mor}(\mathcal{P}_1(M))$ is a smooth space. Clearly $\mathrm{Ob}(\mathcal{P}_1(M))=M$ is a smooth space as well. One can check that $\mathcal{P}_1(M)$ becomes a smooth 2-space with usual composition of paths giving the composition of morphisms.

In fact, $\mathcal{P}_1(M)$ is not just a smooth 2-space: it is also a groupoid. The inverse of $[\gamma]$ is just $[\overline{\gamma}]$, where $\overline{\gamma}$ is obtained by reversing the orientation of the path γ:
$$\overline{\gamma}(s)=\gamma(1-s).$$

Moreover, the map sending any morphism to its inverse is smooth. Thus $\mathcal{P}_1(M)$ is a **smooth groupoid**: a 2-space where every morphism is invertible and the map sending every morphism to its inverse is smooth.

3   2-Bundles

In differential geometry an ordinary bundle consists of two smooth spaces, the **total space** E and the **base space** B, together with a **projection map**

$$E\xrightarrow{\ p\ }B\,.$$

To categorify the theory of bundles, we start by replacing smooth spaces by smooth 2-spaces:

Definition 13. *A **2-bundle** consists of*

- *a smooth 2-space E (the **total 2-space**),*

- *a smooth 2-space B (the **base 2-space**),*

- *a smooth map $p\colon E\to B$ (the **projection**).*

In gauge theory we are interested in *locally trivial* 2-bundles. Ordinarily, a locally trivial bundle with fiber F is a bundle $E\xrightarrow{\ p\ }B$ together with an open cover U_i of B, such that the restriction of E to any of the U_i is equipped with an isomorphism to the trivial bundle $U_i\times F\to U_i$. To categorify this, we would need to define a '2-cover' of the base 2-space B. This is actually a rather tricky issue, since forming the 'union' of 2-spaces requires knowing how to compose a morphism in one 2-space with a morphism in another. While this issue can be addressed, we prefer to avoid it here by assuming that B is just an ordinary smooth space, regarded as a smooth 2-space with only identity morphisms.

We can now state the definition of a locally trivial 2-bundle. First note that we can restrict a 2-bundle $E\xrightarrow{\ p\ }B$ to any subspace $U\subseteq B$ to obtain a 2-bundle which we denote by $E|_U\xrightarrow{\ p\ }U$. Then:

Definition 14. *Given a smooth 2-space F, we define a* **locally trivial 2-bundle with fiber** *F to be a 2-bundle $E \xrightarrow{p} B$ and an open cover $\{U_i\}$ of the base space B equipped with equivalences*

$$E|_{U_i} \xrightarrow{t_i} U_i \times F$$

called **local trivializations** *such that these diagrams:*

commute for all $i \in I$.

Readers wise in the ways of categorification [39] may ask why we did not require that these diagrams commute *up to natural isomorphism*. The reason is that U_i, as an ordinary space, has only identity morphisms when we regard it as a 2-space. Thus, for this diagram to commute up to natural isomorphism, it must commute 'on the nose'.

Readers less wise in the ways of categorification may find the above definition painfully abstract. So, let us translate it into data that specify how to build a locally trivial 2-bundle from trivial ones over the patches U_i. For this, we need to extract *transition functions* from the local trivializations.

So, suppose $E \xrightarrow{p} B$ is a locally trivial 2-bundle with fiber F. This means that B is equipped with an open cover U and for each open set U_i in the cover we have a local trivialization

$$t_i \colon E|_{U_i} \to U_i \times F$$

which is an equivalence. This means that t_i is equipped with a specified map

$$\bar{t}_i \colon U_i \times F \to E|_{U_i}$$

together with invertible 2-maps

$$
\begin{aligned}
\tau_i \colon \bar{t}_i t_i &\Rightarrow 1 \\
\bar{\tau}_i \colon t_i \bar{t}_i &\Rightarrow 1
\end{aligned}
$$

In particular, this means that $\bar{t}_i$ is also an equivalence.

Now consider a double intersection $U_{ij} = U_i \cap U_j$. The composite of equivalences is again an equivalence, so we get an **autoequivalence**

$$t_j \bar{t}_i \colon U_{ij} \times F \to U_{ij} \times F$$

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that is, an equivalence from this 2-space to itself. By the commutative diagram in Def. 14, this autoequivalence must act trivially on the U_{ij} factor, so

$$t_j \bar{t}_i(x, f) = (x, f g_{ij}(x))$$

for some smooth function g_{ij} from U_{ij} to the smooth space of autoequivalences of the fiber F. Note that we write these autoequivalences as acting on F from the right, as customary in the theory of bundles. We call the functions g_{ij} **transition functions**, since they are just categorified versions of the usual transition functions for locally trivial bundles.

In fact, for any smooth 2-space F there is a smooth 2-space $\mathcal{AUT}(F)$ whose objects are autoequivalences of F and whose morphisms are invertible 2-maps between these. The transition functions are maps

$$g_{ij} \colon U_{ij} \to \mathrm{Ob}(\mathcal{AUT}(F)).$$

The 2-space $\mathcal{AUT}(F)$ is a kind of 2-group, with composition of autoequivalences giving the product. However, is not the sort of 2-group we have been considering here, because it does not have 'strict inverses': the group laws involving inverses do not hold as equations, but only up to specified isomorphisms that satisfy coherence laws of their own. So, $\mathcal{AUT}(F)$ is a 'coherent' smooth 2-group in the sense of Baez and Lauda [12].

Next, consider a triple intersection $U_{ijk} = U_i \cap U_j \cap U_k$. In an ordinary locally trivial bundle the transition functions satisfy the equation $g_{ij} g_{jk} = g_{ik}$, but in a locally trivial 2-bundle this holds *only up to isomorphism*. In other words, there is a smooth map

$$h_{ijk} \colon U_{ijk} \to \mathrm{Mor}(\mathcal{AUT}(F))$$

such that for any $x \in U_{ijk}$,

$$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \xrightarrow{\sim} g_{ik}(x).$$

To see this, note that there is an invertible 2-map

$$t_k \tau_j \bar{t}_i \colon t_k \bar{t}_j t_j \bar{t}_i \Rightarrow t_k \bar{t}_i$$

defined by horizontally composing τ_j with t_k on the left and $\bar{t}_i$ on the right. Since

$$t_k \bar{t}_j t_j \bar{t}_i(x, f) = (x, f g_{ij}(x) g_{jk}(x))$$

while

$$t_k \bar{t}_i(x, f) = (x, f g_{ik}(x))$$

we have

$$t_k \tau_j \bar{t}_i(x, f) \colon (x, f g_{ij}(x) g_{jk}(x)) \to (x, f g_{ik}(x)).$$

Since this morphism must be the identity on the first factor, we have

$$t_k \tau_j \bar{t}_i(x, f) = (1_x, f h_{ijk}(x))$$

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where $h_{ijk}(x)\colon g_{ij}(x)g_{jk}(x) \to g_{ik}(x)$ depends smoothly on x.

Similarly, in a locally trivial bundle we have $g_{ii} = 1$, but in a locally trivial 2-bundle there is a smooth map

$$k_i\colon U_i \to \mathrm{Mor}(\mathcal{AUT}(F))$$

such that for any $x \in U_i$,
$$k_i(x)\colon g_{ii}(x) \to 1.$$

To see this, recall that there is an invertible 2-map

$$\bar{\tau}_i\colon t_i\bar{t}_i \Rightarrow 1.$$

Since
$$t_i\bar{t}_i(x, f) = (x, fg_{ii}(x))$$

we have
$$\bar{\tau}_i(x, f)\colon (x, fg_{ii}(x)) \to (x, f),$$

and since this morphism must be the identity on the first factor, we have

$$\bar{\tau}_i(x, f) = (1_x, fk_i(x))$$

where $k_i(x)\colon g_{ii}(x) \to 1$ depends smoothly on x.

In short, the transition functions g_{ij} for a locally trivial 2-bundle satisfy the usual cocycle conditions up to specified isomorphisms h_{ijk} and k_i, which we call **higher transition functions**. These, in turn, satisfy some cocycle conditions of their own:

Theorem 15. *Suppose $E \xrightarrow{p} B$ is a locally trivial 2-bundle, and define the transition functions $g_{ij}, h_{ijk},$ and k_i as above. Then:*

- *h makes this diagram, called the **associative law**, commute for any $x \in U_{ijkl}$:*

$$g_{ij}(x)\, g_{kl}(x)\, g_{lm}(x)$$

$$g_{im}(x)$$

with labels $h_{ijk}(x)\, g_{kl}(x)$, $g_{ij}(x)\, h_{jkl}(x)$, $g_{il}(x)\, g_{lm}(x)$, $g_{ij}(x)\, g_{jm}(x)$, $h_{ikl}(x)$, $h_{ijl}(x)$

- *k makes these diagrams, called the **left and right unit laws**, commute for any $x \in U_{ij}$:*

$$g_{ii}(x)\, g_{ij}(x) \xrightarrow{k_i(x)\, g_{ij}(x)} 1\, g_{ij}(x)$$

$$h_{iij}(x) \searrow \qquad \Big\| =$$

$$g_{ij}(x)$$

$$g_{ij}(x)\, 1 \xleftarrow{g_{ij}(x)\, k_j(x)} g_{ij}(x)\, g_{jj}(x)$$

$$= \Big\| \qquad \swarrow h_{ijj}(x)$$

$$g_{ij}(x)$$

Proof. Checking that these diagrams commute is a straightforward computation using the definitions of g, h, and k in terms of $t, \bar{t}, \tau$ and $\bar{\tau}$. $\square$

The associative law and unit laws are analogous to those which hold in a monoid. They also have simplicial interpretations. In a locally trivial bundle, the transition functions give a commuting triangle for any triple intersection:
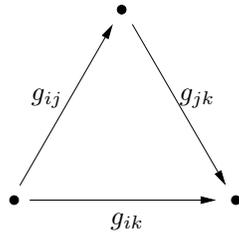
In a locally trivial 2-bundle, such triangles commute only up to isomorphism:
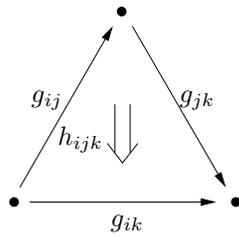
However, the associative law says that for each quadruple intersection, this tetrahedron commutes:
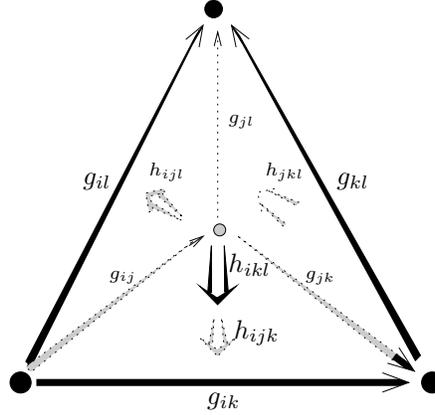
We can also visualize the left and right unit laws simplicially, but they involve degenerate tetrahedra:
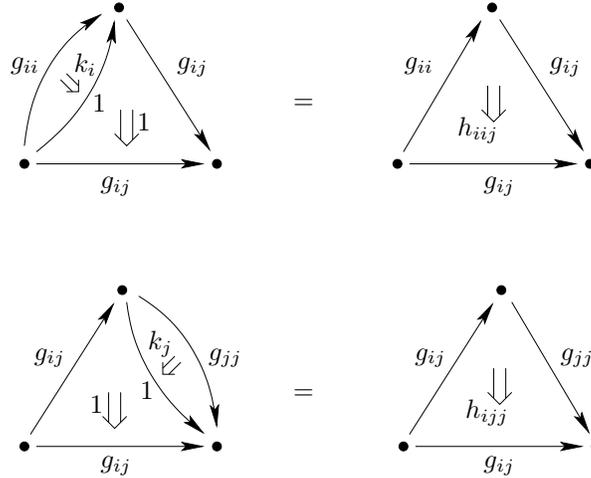
We are now almost in a position to define $\mathcal{G}$-2-bundles for any smooth 2-group $\mathcal{G}$; we only need to understand how a 2-group can 'act' on a 2-space. For simplicity we only consider the case of a strict action:

Definition 16. *A* **(strict) action** *of a smooth 2-group* $\mathcal{G}$ *on a smooth 2-space* F *is a smooth homomorphism*

$$\alpha : \mathcal{G} \to \mathcal{AUT}(F),$$

that is, a smooth map that preserves products and inverses.

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Note in particular that every smooth 2-group has an action on itself via right multiplication.

Definition 17. *For any smooth 2-group $\mathcal{G}$, we say a locally trivial 2-bundle $E \to M$ has $\mathcal{G}$ as its* **structure 2-group** *when the transition functions g_{ij}, h_{ijk}, and k_i factor through an action $\mathcal{G} \to \mathcal{AUT}(F)$. In this case we also say P is a $\mathcal{G}$-**2-bundle***. If furthermore $F = \mathcal{G}$ and $\mathcal{G}$ acts on F by right multiplication, we say P is a* **principal** $\mathcal{G}$-**2-bundle**.

For a principal $\mathcal{G}$-2-bundle we can think of the transition functions as taking values in the groups $\mathrm{Ob}(\mathcal{G})$ and $\mathrm{Mor}(\mathcal{G})$. The reader familiar with gerbes will note that these functions, satisfying the equations they do, reduce to the usual sort of cocycle defining an *abelian gerbe* when $k_i = 1$ and $\mathcal{G}$ has the special form described in Example 7. Similarly, they reduce to a cocycle defining a *nonabelian gerbe* when $k_i = 1$ and $\mathcal{G}$ has the form described in Example 8. Thus there is a close relation between principal 2-bundles and gerbes, much like that between principal bundles and sheaves of groups.

The equation $k_i = 1$ arises because gerbes are often defined using Čech cocycles that are antisymmetric in the indices $i, j, k, \ldots$, in the sense that group-valued functions go to their inverses upon an odd permutation of these indices. Thus in this context $g_{ii} = 1$, and one implicitly assumes $k_i = 1$. In fact, Bartels [17] has shown that every $\mathcal{G}$-2-bundle is equivalent to one with $k_i = 1$. To state this result, he first needed to define a 2-category of $\mathcal{G}$-2-bundles. This 2-category is equivalent to the 2-category of abelian or nonabelian gerbes when $\mathcal{G}$ has one of the two special forms mentioned above.

4   2-Connections

For a trivial bundle, the holonomy of a connection assigns elements of the structure group to paths in space. Similarly, a 2-connection assigns objects and morphisms of the structure 2-group to paths and surfaces in space. To make this precise we need the notion of a 'path 2-groupoid'.

We described the path groupoid of a smooth space M in Example 12. This has points of M as objects:

$$\bullet\, x$$

and thin homotopy classes of paths in M as morphisms:

The path 2-groupoid also has 2-morphisms, which are thin homotopy classes of 2-dimensional surfaces like this:

We call these 'bigons':

Definition 18. *Given a smooth space M, a* **parametrized bigon** *in M is a smooth map*

$$\Sigma \colon [0,1]^2 \to M$$

which is constant near $s = 0$, constant near $s = 1$, independent of t near $t = 0$, and independent of t near $t = 1$. We call $\Sigma(\cdot, 0)$ the **source** *of the parametrized bigon Σ, and $\Sigma(\cdot, 1)$ the* **target**. *If Σ is a parametrized bigon with source γ_1 and target γ_2, we write $\Sigma \colon \gamma_1 \to \gamma_2$.*

Definition 19. *Suppose $\Sigma \colon \gamma_1 \to \gamma_2$ and $\Sigma' \colon \gamma_1' \to \gamma_2'$ are parametrized bigons in a smooth space M. A* **thin homotopy** *between Σ and Σ' is a smooth map*

$$H \colon [0,1]^3 \to M$$

with the following properties:

- *$H(s,t,0) = \Sigma(s,t)$ and $H(s,t,1) = \Sigma'(s,t)$,*

- *$H(s,t,u)$ is independent of u near $u = 0$ and near $u = 1$,*

- *For some thin homotopy F_1 from γ_1 to γ_1', $H(s,t,u) = F_1(s,u)$ for t near 0, and for some thin homotopy F_2 from γ_2 to γ_2', $H(s,t,u) = F_2(s,u)$ for t near 1,*

- *$H(s,t,u)$ is constant for $s = 0$ and near $s = 1$,*

- *H does not sweep out any volume: the rank of the differential $dH(s,t,u)$ is < 3 for all $s,t,u \in [0,1]$.*

We say two parametrized bigons Σ, Σ' lie in the same **thin homotopy class** *if the pair (Σ, Σ') lies in the closure of the thin homotopy equivalence relation. A* **bigon** *is a thin homotopy class $[\Sigma]$ of parametrized bigons.*

Definition 20. *The* **path 2-groupoid** *$\mathcal{P}_2(M)$ of a smooth space M is the 2-category in which:*

- *objects are points $x \in M$:*

$$\bullet\, x$$

- *morphisms are thin homotopy classes of paths γ in M that are constant near $s = 0$ and $s = 1$:*

- *2-morphisms are bigons in M*

and whose composition operations are defined as:

- $$x \xrightarrow{[\gamma_1]} y \xrightarrow{[\gamma_2]} z \;=\; x \xrightarrow{[\gamma_1 \circ \gamma_2]} z$$

  where

  $$(\gamma_1 \circ \gamma_2)(s) := \begin{cases} \gamma_1(2s) & \text{for } 0 \le s \le 1/2 \\ \gamma_2(2s-1) & \text{for } 1/2 \le s \le 1 \end{cases}$$

- $$x \Downarrow \begin{smallmatrix}[\gamma_1]\\ [\Sigma_1]\\ [\gamma_2] \\ [\Sigma_2] \\ [\gamma_3]\end{smallmatrix} y \;=\; x \Downarrow \begin{smallmatrix}[\gamma_1]\\ [\Sigma_1\Sigma_2]\\ [\gamma_3]\end{smallmatrix} y$$

  where

  $$(\Sigma_1\Sigma_2)(s,t) := \begin{cases} \Sigma_1(s, 2t) & \text{for } 0 \le t \le 1/2 \\ \Sigma_2(s, 2t-1) & \text{for } 1/2 \le t \le 1 \end{cases}$$

- $$x \Downarrow \begin{smallmatrix}[\gamma_1]\\ [\Sigma_1]\\ \gamma_1'\end{smallmatrix} y \Downarrow \begin{smallmatrix}[\gamma_2]\\ [\Sigma_2]\\ [\gamma_2']\end{smallmatrix} z \;=\; x \Downarrow \begin{smallmatrix}[\gamma_1 \circ \gamma_2]\\ [\Sigma_1 \circ \Sigma_2]\\ [\gamma_1' \circ \gamma_2']\end{smallmatrix} z$$

  where

  $$(\Sigma_1 \circ \Sigma_2)(s,t) := \begin{cases} \Sigma_1(2s, t) & \text{for } 0 \le s \le 1/2 \\ \Sigma_2(2s-1, t) & \text{for } 1/2 \le s \le 1 \end{cases}$$

One can check that these operations are well-defined, where for vertical composition we must choose suitable representatives of the bigons being composed. One can also check that $\mathcal{P}_2(M)$ is indeed a 2-category. Furthermore, the objects, morphisms and 2-morphisms in $\mathcal{P}_2(M)$ all form smooth spaces, by an elaboration of the ideas in Example 12, and all the 2-category operations are then smooth maps. We thus say $\mathcal{P}_2(M)$ is a **smooth 2-category**: that is, a 2-category in C^∞. Indeed, the usual definitions [40] of 2-category, 2-functor, pseudonatural transformation, and modification can all be internalized in C^∞, and we use these 'smooth' notions in what follows. Furthermore, both morphisms and 2-morphisms in $\mathcal{P}_2(M)$ have strict inverses, and the operations of taking inverses are smooth, so we say $\mathcal{P}_2(M)$ is a **smooth 2-groupoid**.

We obtain the notion of '2-connection' by categorifying the concept of connection. The following result suggests a strategy for doing this:

Theorem 21. *For any smooth group G and smooth space B, suppose $E \to B$ is a principal G-bundle equipped with local trivializations over open sets $\{U_i\}_{i \in I}$ covering B. Let g_{ij} be the transition functions. Then there is a one-to-one correspondence between connections on E and data of the following sort:*

- *for each $i \in I$ a smooth map between smooth 2-spaces:*

$$\mathrm{hol}_i \colon \mathcal{P}_1(U_i) \to G$$

  called the **local holonomy functor***, from the path groupoid of U_i to the group G regarded as a smooth 2-space with a single object $\bullet$,*

such that:

- *for each $i, j \in I$, the transition function g_{ij} defines a smooth natural isomorphism:*

$$\mathrm{hol}_i|_{U_{ij}} \xrightarrow{\ g_{ij}\ } \mathrm{hol}_j|_{U_{ij}}$$

  called the **transition natural isomorphism***. In other words, this diagram commutes:*

for any path $\gamma \colon x \to y$ in U_{ij}.

Proof. See Baez and Schreiber [23]. □

In addition, it is worth noting that whenever we have a connection, for each $i, j, k \in I$ this triangle commutes:

The idea behind the above result is that:

- The local holonomy functors hol_i are specified by 1-forms

$$A_i \in \Omega^1(U_i, \mathfrak{g}) \ .$$

- The transition natural isomorphisms g_{ij} are specified by smooth functions

$$g_{ij} \colon U_{ij} \to G \ ,$$

satisfying the equation

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} \mathbf{d} g_{ij}^{-1}$$

on U_{ij}.

- The commuting triangle for the triple intersection U_{ijk} is equivalent to the equation

$$g_{ij} g_{jk} = g_{ik}$$

on U_{ijk}.

Categorifying all this, we make the following definition:

Definition 22. *For any smooth 2-group $\mathcal{G}$, suppose that $E \to B$ is a principal $\mathcal{G}$-2-bundle equipped with local trivializations over open sets $\{U_i\}_{i \in I}$ covering B, and let the transition functions g_{ij}, h_{ijk} and k_i be given as in Theorem 15. Suppose for simplicity that $k_i = 1$. Then a **2-connection** on E consists of the following data:*
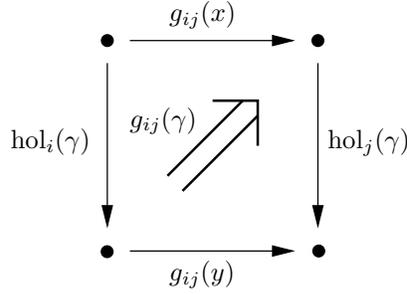
- *for each $i \in I$ a smooth 2-functor*

*called the **local holonomy 2-functor**, from the path 2-groupoid $\mathcal{P}_2(U_i)$ to the 2-group $\mathcal{G}$ regarded as a smooth 2-category with a single object $\bullet$,*
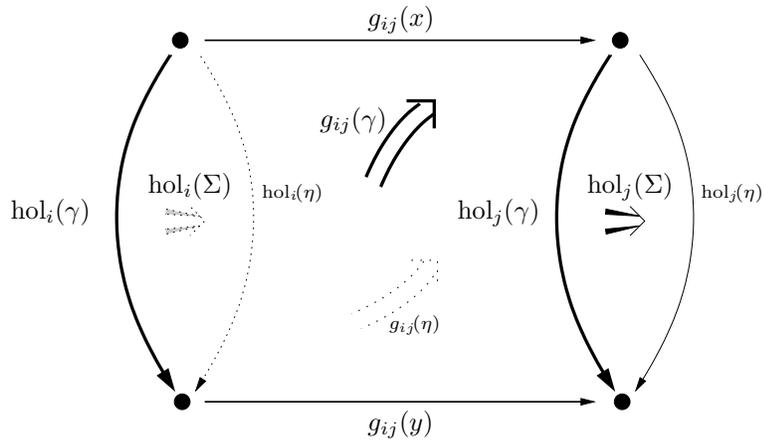
such that:

- *For each i, j a pseudonatural isomorphism:*

$$g_{ij} \colon \mathrm{hol}_i|_{\mathcal{P}(U_i \cap U_j)} \to \mathrm{hol}_j|_{\mathcal{P}(U_i \cap U_j)}$$

extending the transition function g_{ij}. In other words, for each path $\gamma \colon x \to y$ in $U_i \cap U_j$ a morphism in G:
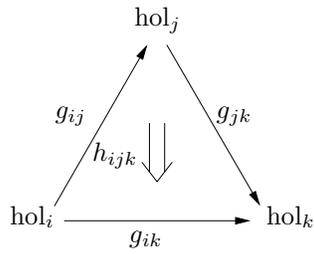
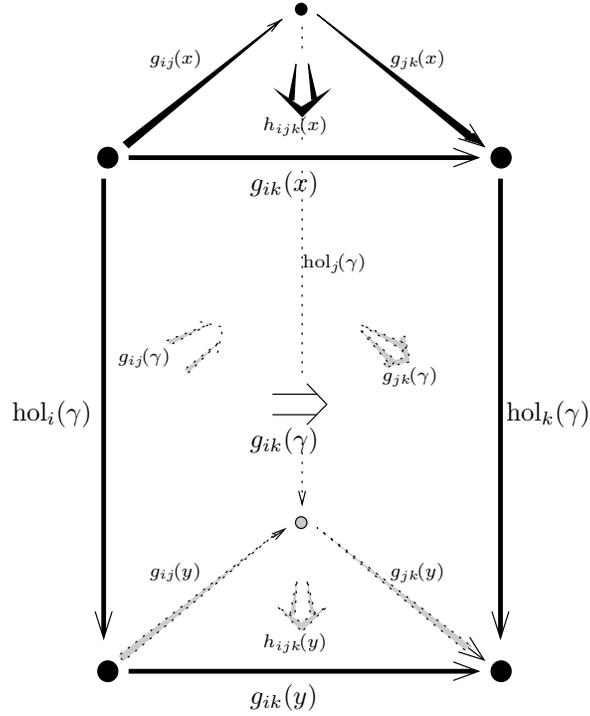*depending smoothly on γ, such that this diagram commutes:*

for any bigon $\Sigma\colon \gamma \Rightarrow \eta$ in U_{ij},

- *for each $i, j, k \in I$ the transition function h_{ijk} defines a modification:*
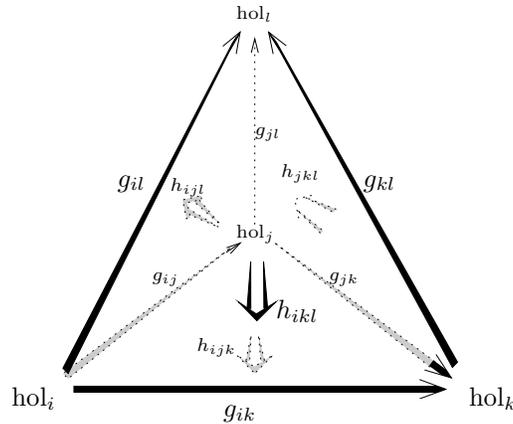

In other words, this diagram commutes:

for any bigon $\Sigma\colon \gamma \Rightarrow \eta$ in U_{ijk}.

In addition, it is worth noting that whenever we have a 2-connection, for each $i, j, k, l \in I$ this tetrahedron commutes:

In analogy to the situation for ordinary connections on bundles, one would like to obtain 2-connections from Lie-algebra-valued differential forms. This is our next result. In what follows, (G, H, t, α) will be the smooth crossed module

corresponding to the smooth 2-group $\mathcal{G}$. We think of the transition function g_{ij} as taking values in $\mathrm{Ob}(\mathcal{G}) = G$, and think of h_{ijk} as taking values in H. Actually h_{ijk} takes values in $\mathrm{Mor}(\mathcal{G}) \cong G \ltimes H$, but its G component is determined by its source, so only its H component is interesting. In these terms, the fact that

$$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \xrightarrow{\sim} g_{ik}(x)$$

translates into the equation

$$g_{ij}(x)\, g_{jk}(x)\, t(h_{ijk}) = g_{ik}(x),$$

and the associative law of Theorem 15 (i.e. the above tetrahedron) becomes a cocycle condition familiar from the theory of nonabelian gerbes:

$$h_{ijk}\, h_{ikl} = \alpha(g_{ij})(h_{jkl})\, h_{ijl}\,.$$

Theorem 23. *For any smooth 2-group $\mathcal{G}$ and smooth space B, suppose that $E \to B$ is a principal $\mathcal{G}$-2-bundle equipped with local trivializations over open sets $\{U_i\}_{i \in I}$ covering B, with the transition functions g_{ij}, h_{ijk} and k_i given as in Theorem 15. Suppose for simplicity that $k_i = 1$. Let (G, H, t, α) be the smooth crossed module corresponding to $\mathcal{G}$, and let $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ be the corresponding differential crossed module. Then there is a one-to-one correspondence between 2-connections on E and Lie-algebra-valued differential forms (A_i, B_i, a_{ij}) satisfying certain equations, as follows:*

- *The local holonomy 2-functor hol_i is specified by differential forms*

$$A_i \in \Omega^1(U_i, \mathfrak{g})$$
$$B_i \in \Omega^2(U_i, \mathfrak{h})$$

 satisfying

$$F_{A_i} + dt(B_i) = 0\,,$$

 where $F_{A_i} = \mathbf{d}A_i + A_i \wedge A_i$ is the curvature 2-form of A_i.

- *The transition pseudonatural isomorphism $\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$ is specified by the transition functions g_{ij} together with differential forms*

$$a_{ij} \quad \in \quad \Omega^1(U_{ij}, \mathfrak{h})$$

 satisfying the equations:

$$
\begin{aligned}
A_i &= g_{ij} A_j g_{ij}^{-1} + g_{ij}\mathbf{d}g_{ij}^{-1} - dt(a_{ij}) \\
B_i &= \alpha(g_{ij})(B_j) + k_{ij}
\end{aligned}
$$

 on U_{ij}, where

$$k_{ij} = \mathbf{d}a_{ij} + a_{ij} \wedge a_{ij} + d\alpha(A_i) \wedge a_{ij}\,.$$

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- *The modification $g_{ij} \circ g_{jk} \xrightarrow{h_{ijk}} g_{ik}$ is specified by the transition functions h_{ijk}. For this, the differential forms a_{ij} are required to satisfy the equation:*

$$a_{ij} + \alpha(g_{ij})a_{jk} = h_{ijk}a_{ik}h_{ijk}^{-1} + (\mathbf{d}h_{ijk})h_{ijk}^{-1} + d\alpha(A_i)(h_{ijk})\,h_{ijk}^{-1}$$

  on U_{ijk}.

Proof. See Baez and Schreiber [23]. The 'vanishing fake curvature' condition $F_{A_i} + dt(B_i) = 0$ is necessary for the holonomy 2-functor to preserve the source and target of 2-morphisms. It also guarantees that the holonomy over a parametrized bigon is invariant under thin homotopies. □

The reader familiar with gerbes will recognize that Lie-algebra-valued differential forms of the above sort give a *connection on an abelian gerbe* when $\mathcal{G}$ is of the special form described in Example 7. Similarly, they give rise to a *connection with vanishing fake curvature on a nonabelian gerbe* when $\mathcal{G}$ is of the form described in Example 8.

The vanishing fake curvature condition is a strong one. As Breen has emphasized, it implies that the $\mathfrak{h}$-valued 'curvature' 3-form $H = \mathbf{d}B + d\alpha(A) \wedge B$ actually takes values in the kernel of dt, which is an abelian ideal of $\mathfrak{h}$. So, the existence of well-behaved holonomies forces a 2-connection to be somewhat abelian in nature.

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