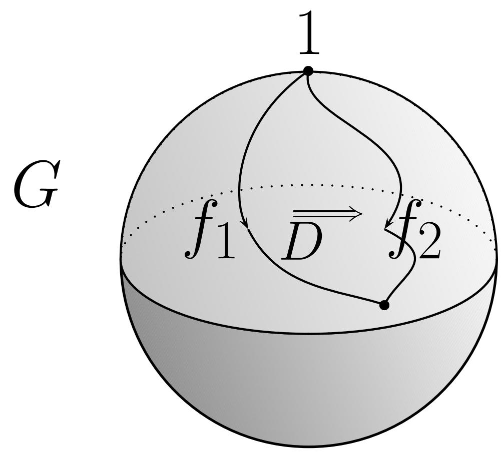
Higher Gauge Theory
and the String Group

John C. Baez

joint work with Toby Bartels, Alissa Crans, Aaron Lauda,
Urs Schreiber, and Danny Stevenson

Erwin Schrödinger Institute
August 22, 2007

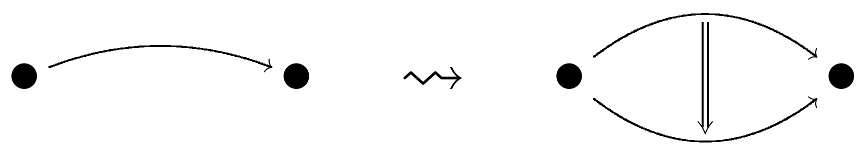


For more see: http://math.ucr.edu/home/baez/esi/

Categorification

sets $\rightsquigarrow$ **categories**
functions $\rightsquigarrow$ **functors**
equations $\rightsquigarrow$ **natural isomorphisms**

Categorification 'boosts the dimension' by one:

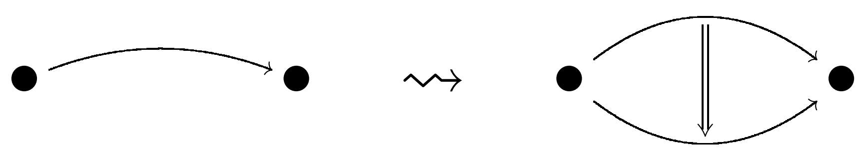


In **strict** categorification we keep equations as equations. This is evil... but today we'll do it whenever it doesn't cause trouble, just to save time.

Higher Gauge Theory

$$\begin{aligned} \textbf{groups} &\rightsquigarrow \textbf{2-groups} \\ \textbf{Lie algebras} &\rightsquigarrow \textbf{Lie 2-algebras} \\ \textbf{bundles} &\rightsquigarrow \textbf{2-bundles} \\ \textbf{connections} &\rightsquigarrow \textbf{2-connections} \end{aligned}$$

Connections describe parallel transport for particles.
2-Connections describe parallel transport for strings!



We should even go beyond $n = 2$... but not today.

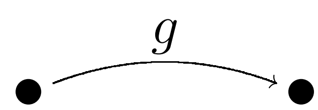
Fix a simply-connected compact simple Lie group G. Then:

- The Lie algebra $\mathfrak{g}$ gives a 1-parameter family of Lie 2-algebras $\mathfrak{string}_k(\mathfrak{g})$.
- When $k \in \mathbb{Z}$, $\mathfrak{string}_k(\mathfrak{g})$ comes from a Lie 2-group $\mathrm{String}_k(G)$.
- The 'geometric realization of the nerve' of $\mathrm{String}_k(G)$ is a topological group, $|\mathrm{String}_k(G)|$.
- Principal $\mathrm{String}_k(G)$-2-bundles are the same as $|\mathrm{String}_k(G)|$-bundles.
- For $k = 1$, $|\mathrm{String}_k(G)|$ is G with its 3rd homotopy group made trivial.
- We can define connections and characteristic classes for $\mathrm{String}_k(G)$-2-bundles!

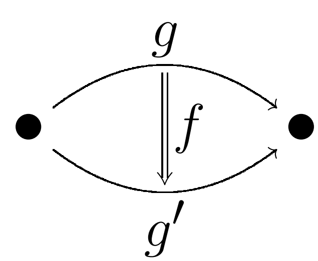
2-Groups

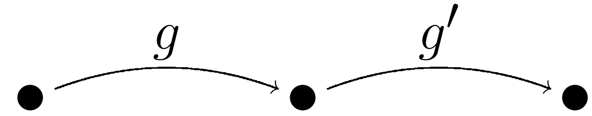
A **strict 2-group** is a category in Grp: a category with a group of objects and a group of morphisms, such that all the category operations are group homomorphisms.

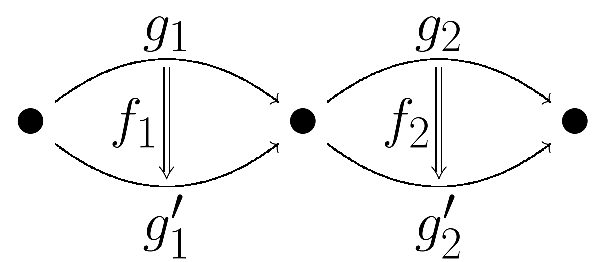
The objects in a 2-group look like this:

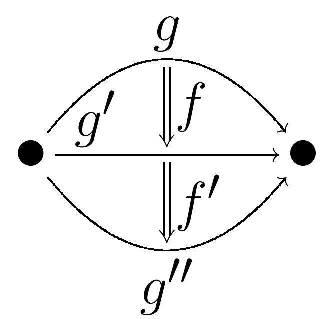


The morphisms look like this:

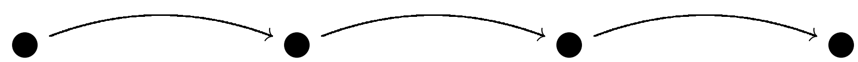


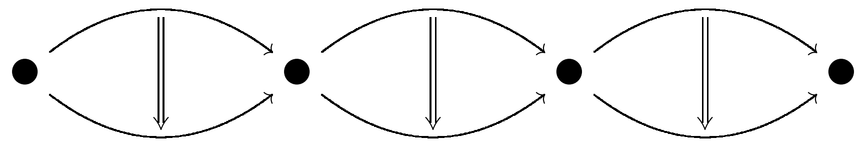
We can multiply objects: 

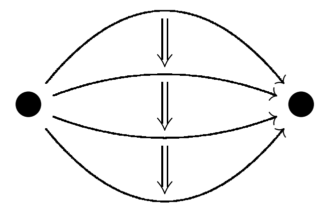
multiply morphisms: 

and compose morphisms: 

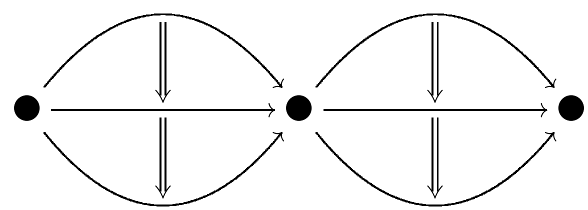
All 3 operations have a unit and inverses. All 3 are associative, so these are well-defined:







Finally, the **interchange law** holds, meaning



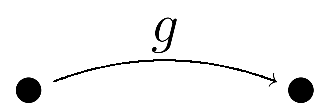
is well-defined.

Mac Lane and Whitehead first introduced 2-groups in the disguise of 'crossed modules':

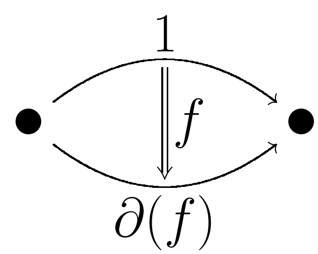
$$G_0 \xleftarrow{\partial} G_1$$

Here G_0 and G_1 are groups, and G_0 acts on G_1 in a manner compatible with the differential ∂.

To get a crossed module from a 2-group, just let G_0 be the group of objects:



and G_1 be the group of morphisms starting at 1. The differential ∂ is defined as follows:



Lie 2-Algebras

A **strict Lie 2-algebra** is a category in LieAlg: a category with a Lie algebra of objects and a Lie algebra of morphisms, such that all the category operations are Lie algebra homomorphisms.

A strict Lie 2-algebra can be viewed as an 'infinitesimal crossed module':

$$\mathfrak{g}_0 \xleftarrow{\partial} \mathfrak{g}_1$$

Here $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are Lie algebras, and $\mathfrak{g}_0$ acts as derivations of $\mathfrak{g}_1$ in a manner compatible with the differential ∂.

Theorem (Mac Lane, Sinh). A 2-group is determined up to equivalence by:

- the group G of isomorphism classes of objects,
- the abelian group A of endomorphisms of any object,
- an action of G on A,
- an element of $H^3(G, A)$.

Theorem (Gerstenhaber, Crans). A Lie 2-algebra is determined up to equivalence by:

- the Lie algebra $\mathfrak{g}$ of isomorphism classes of objects,
- the vector space $\mathfrak{a}$ of endomorphisms of any object,
- a representation of $\mathfrak{g}$ on $\mathfrak{a}$,
- an element of $H^3(\mathfrak{g}, \mathfrak{a})$.

Suppose G is a simply-connected compact simple Lie group. Let $\mathfrak{g}$ be its Lie algebra. A lemma of Whitehead says:

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

So:

Corollary. For any $k \in \mathbb{R}$ there is a Lie 2-algebra $\mathfrak{string}_k(\mathfrak{g})$ for which:

- $\mathfrak{g}$ is the Lie algebra of isomorphism classes of objects;
- $\mathbb{R}$ is the vector space of endomorphisms of any object.

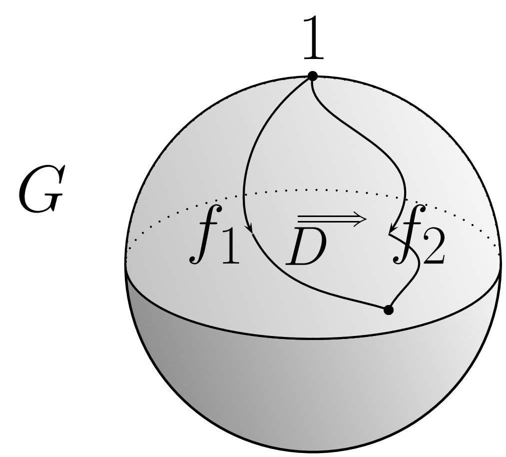
Every Lie 2-algebra with these properties is equivalent to $\mathfrak{string}_k(\mathfrak{g})$ for some unique $k \in \mathbb{R}$.

Theorem. For any $k \in \mathbb{Z}$, $\mathfrak{string}_k(\mathfrak{g})$ is the Lie 2-algebra of an infinite-dimensional Lie 2-group $\mathrm{String}_k(G)$.

An object of $\mathrm{String}_k(G)$ is a smooth path

$$f\colon [0, 2\pi] \to G$$

starting at the identity. A morphism from f_1 to f_2 is an equivalence class of pairs (D, α) where D is a disk going from f_1 to f_2 and $\alpha \in \mathrm{U}(1)$:



Any two such pairs (D_1, α_1) and (D_2, α_2) have a 3-ball B whose boundary is $D_1 \cup D_2$. The pairs are equivalent when

$$\exp\left(2\pi i k \int_B \nu\right) = \alpha_2/\alpha_1$$

where ν is the left-invariant closed 3-form on G with

$$\nu(x, y, z) = \langle [x, y], z\rangle$$

and $\langle \cdot, \cdot \rangle$ is the smallest invariant inner product on $\mathfrak{g}$ such that ν gives an integral cohomology class.

Theorem. The morphisms in $\mathrm{String}_k(G)$ starting at the constant path form the level-k central extension of the loop group ΩG:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \longrightarrow \Omega G \longrightarrow 1$$

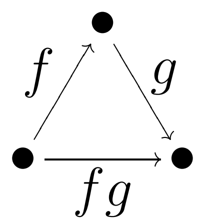
For any category $\mathcal{C}$ there is a space $|\mathcal{C}|$, the **geometric realization of the nerve** of $\mathcal{C}$, built from a vertex for each object:

$$\bullet\ x$$

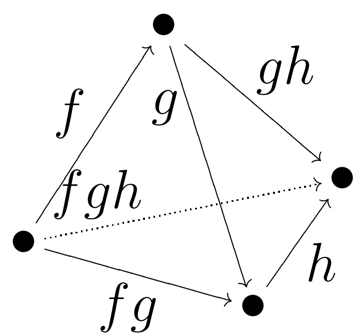
an edge for each morphism:

$$\bullet \xrightarrow{f} \bullet$$

a triangle for each composable pair of morphisms:



a tetrahedron for each composable triple:



and so on...

A 2-group is a category *with a product and inverses.* So, if $\mathcal{G}$ is a 2-group, $|\mathcal{G}|$ is a topological group.

More generally, we can define a topological group $|\mathcal{G}|$ for any *topological* 2-group $\mathcal{G}$.

Theorem. For any $k \in \mathbb{Z}$, there is a short exact sequence of topological groups

$$1 \longrightarrow K(\mathbb{Z},2) \longrightarrow |\mathrm{String}_k(G)| \xrightarrow{\ p\ } G \longrightarrow 1$$

where p is a fibration. Using this we can show:

$$\begin{aligned} \pi_1(|\mathrm{String}_k(G)|) &= 0 \\ \pi_2(|\mathrm{String}_k(G)|) &= \mathbb{Z}/k\mathbb{Z} \\ \pi_3(|\mathrm{String}_k(G)|) &= 0 \qquad\qquad \text{if } k \neq 0 \end{aligned}$$

Theorem. When $k = 1$, $|\mathrm{String}_k(G)|$ is the '3-connected cover' of G: the topological group formed by making the 3rd homotopy group of G trivial.

For example, start with $\mathrm{O}(n)$:

- Making π_0 trivial gives $\mathrm{SO}(n)$.
- Making π_1 trivial gives $\mathrm{Spin}(n)$.
- π_2 of $\mathrm{Spin}(n)$ is already trivial.
- Making π_3 trivial gives $\mathrm{String}(n)$.

We are claiming

$$\mathrm{String}(n) \simeq |\mathrm{String}_k(G)|$$

where $G = \mathrm{Spin}(n)$ and $k = 1$.

2-Bundles — Quick and Dirty

For any topological 2-group $\mathcal{G}$ and any space X, we can define a **principal $\mathcal{G}$-2-bundle over** X to consist of:

- an open cover U_i of X,
- continuous maps

$$g_{ij} \colon U_i \cap U_j \to \mathrm{Ob}(\mathcal{G})$$

  satisfying $g_{ii} = 1$, and

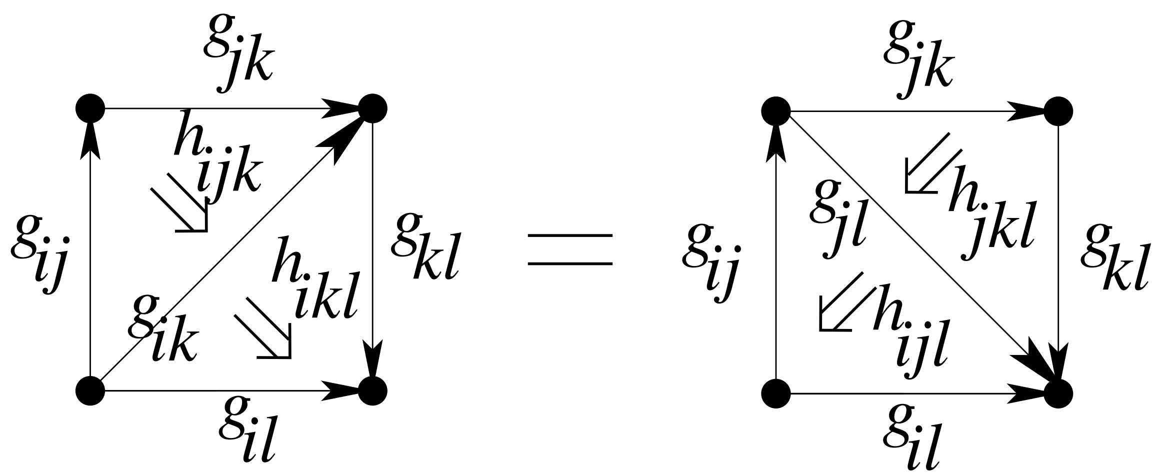
- continuous maps

$$h_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{Mor}(\mathcal{G})$$

  with

$$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \to g_{ik}(x)$$

satisfying the **nonabelian 2-cocycle condition:**



on any quadruple intersection $U_i \cap U_j \cap U_k \cap U_\ell$.

There's a natural notion of 'equivalence' for 2-bundles over X, since they form a 2-category.

Theorem. For any topological 2-group $\mathcal{G}$ and paracompact Hausdorff space X, there is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over X,
- isomorphism classes of principal $|\mathcal{G}|$-bundles over X,
- homotopy classes of maps $f\colon X \to B|\mathcal{G}|$.

So, $B|\mathcal{G}|$ is the classifying space for $\mathcal{G}$-2-bundles.

We have homomorphisms

$$\mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow \mathrm{O}(n)$$

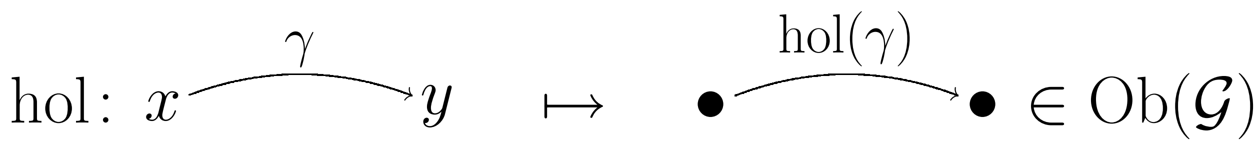
Given an n-dimensional Riemannian manifold X, we can reduce the structure group of the frame bundle from $\mathrm{O}(n)$ to:

- $\mathrm{SO}(n)$ if we have an orientation on X,
- $\mathrm{Spin}(n)$ if we have a spin structure on X,
- $\mathrm{String}(n)$ if we have a string structure on X.

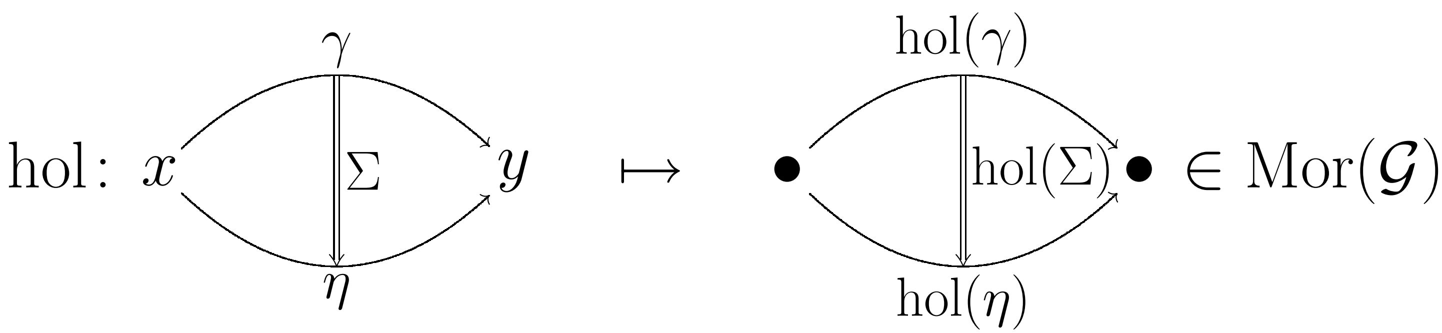
Corollary. For any Riemannian n-manifold X, a string structure on X gives a $\mathcal{G}$-2-bundle over X, where $\mathcal{G} = \mathrm{String}_k(G)$ with $G = \mathrm{Spin}(n)$ and $k = 1$.

2-Connections — Quick and Dirty

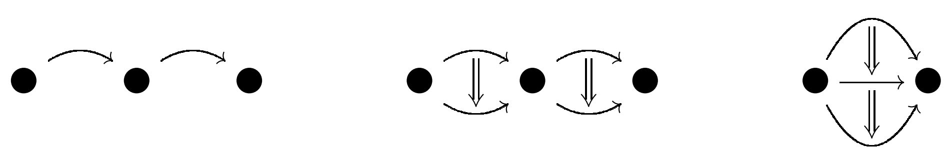
Let $\mathcal{G}$ be a Lie 2-group, P the trivial principal $\mathcal{G}$-2-bundle over some smooth manifold X. A **2-connection** on P assigns holonomies to paths in X:



and surfaces going between paths:



in a manner preserving all 3 forms of composition:



Theorem. Let

$$\mathfrak{g}_0 \xleftarrow{\partial} \mathfrak{g}_1$$

be the infinitesimal crossed module obtained by differentiating the crossed module

$$G_0 \xleftarrow{\partial} G_1$$

corresponding to $\mathcal{G}$. Then there is a 1-1 correspondence between 2-connections on $P \to X$ and **connections**:

- a $\mathfrak{g}_0$-valued 1-form A on X
- a $\mathfrak{g}_1$-valued 2-form B on X

satisfying the **fake flatness** condition:

$$dA + \frac{1}{2}[A, A] + \partial B = 0$$

All this generalizes to nontrivial 2-bundles.

Nice Problem. When $\mathcal{G} = \mathrm{String}_k(G)$, compute the real characteristic classes of a $\mathcal{G}$-2-bundle in terms of an arbitrary connection on this 2-bundle.

The homomorphism $|\mathcal{G}| \xrightarrow{p} G$ gives an algebra homomorphism:

$$H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|\mathcal{G}|, \mathbb{R})$$

When $k = 1$ this is onto, with kernel generated by the 'second Chern class' $c_2 \in H^4(BG, \mathbb{R})$.

In this case, the real characteristic classes of $\mathcal{G}$-2-bundles are just like those of G-bundles, but with the second Chern class killed!