SPACETIME GEOMETRY
AND
CARTAN CONNECTIONS

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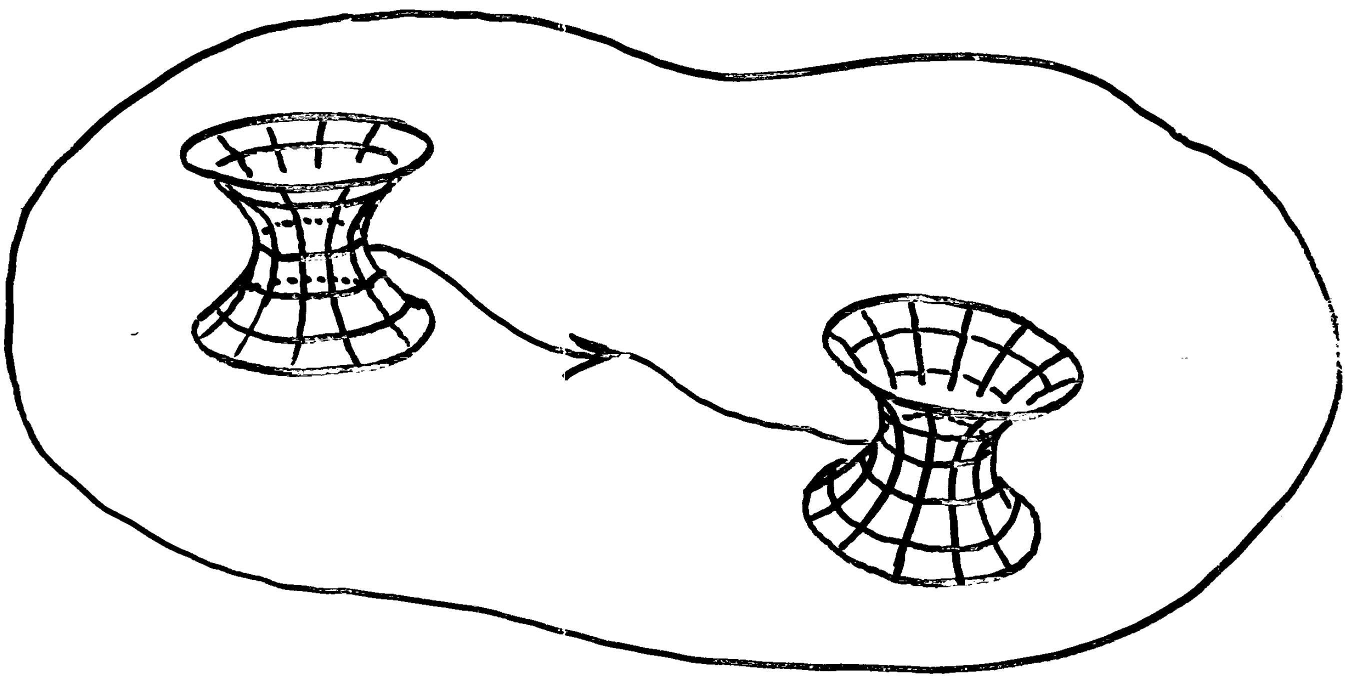


23rd Pacific Coast Gravity Meeting, Caltech
March 16, 17, 2007

MacDowell-Mansouri Gravity

MacDowell & Mansouri (PRL 38 (1977)) found an action for GR with cosmological constant as an SO(4,1) gauge theory:

$$S_{MM} \propto \int F^{ij} \wedge F^{kl} \varepsilon_{ijkl5}$$

$F = dA + A \wedge A$
curv. of SO(4,1) connection

breaks gauge symmetry down to SO(3,1)

How does this give GR?

$$\mathfrak{so}(4,1) = \mathfrak{so}(3,1) \oplus \mathbb{R}^{3,1}$$

gives:

$$A = \underbrace{\omega}_{\text{SO(3,1) conn.}} + \sqrt{\frac{\Lambda}{3}}\, \underbrace{e}_{\text{coframe field}}$$

$$\Rightarrow F = \underbrace{R + \frac{\Lambda}{3} e \wedge e}_{\text{curvature} + \Lambda\text{ term}} + \underbrace{d_\omega e}_{\text{torsion}}$$

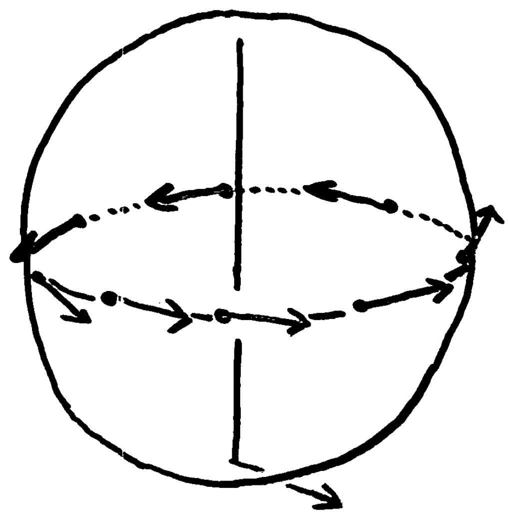
Using this, $S_{MM} = S_{Palatini} + \left(\begin{matrix}\text{topological} \\ \text{term}\end{matrix}\right)$

What's the geometric meaning of breaking the symmetry from $SO(4,1)$ to $SO(3,1)$?

Easier: break from $SO(3)$ to $SO(2)$.

$$\mathfrak{so}(3) = \mathfrak{so}(2) \oplus \mathbb{R}^2$$

tiny rotation of a sphere — tiny rotation fixing chosen basepoint — tiny rotation moving basept.

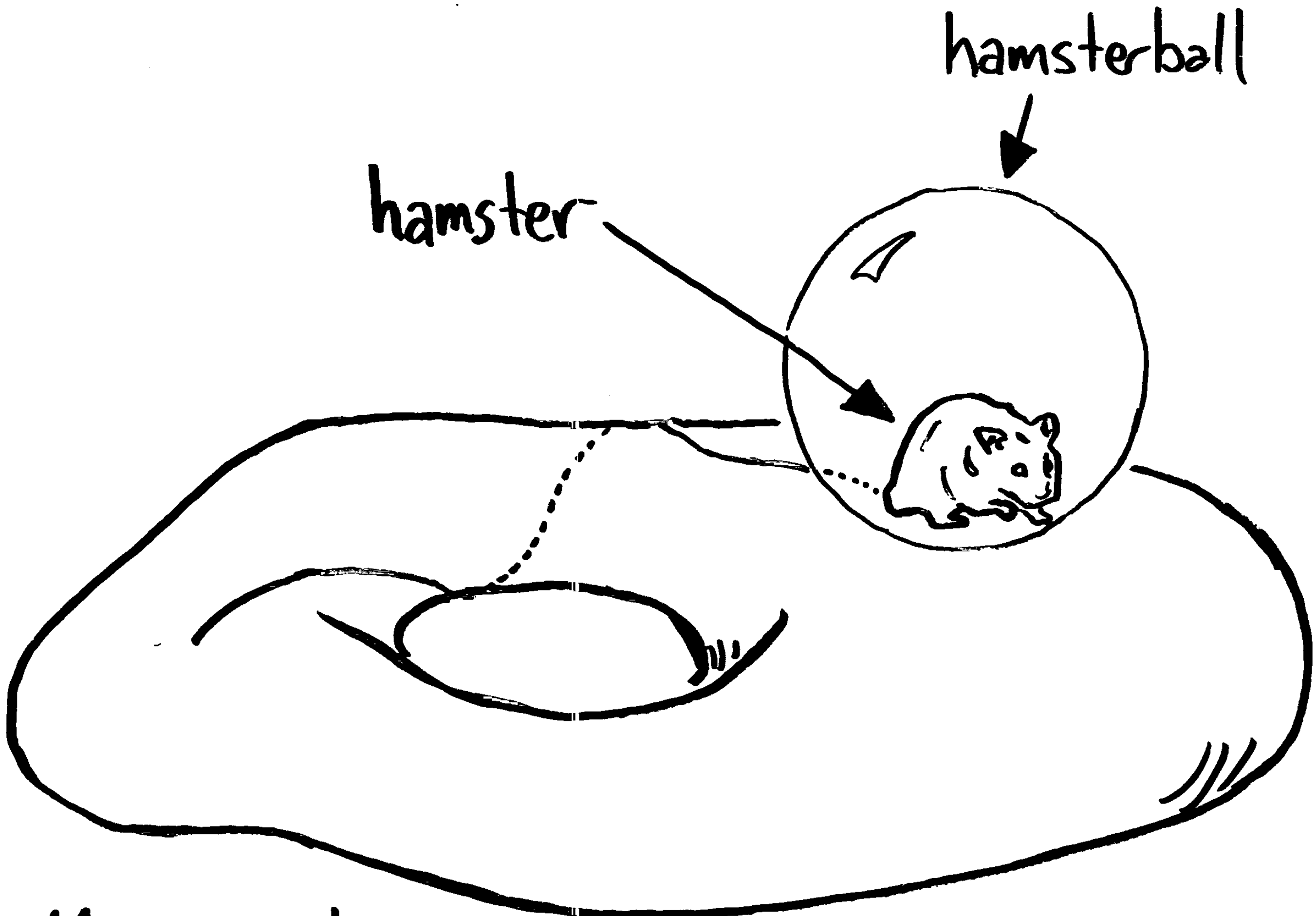


CLAIM: The $SO(3)/SO(2)$-analog of the MacDowell-Mansouri connection is perfect for describing

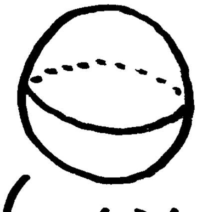
rolling a ball on a surface w/o slipping!

In fact, connections of this type were invented by Cartan in the 1920s, but they often haven't seen the appreciation they deserve!

To understand SO(3)/SO(2) Cartan connections (and rolling w/o slipping) it is helpful to introduce an "observer"...



Main point:

Motion of  is <u>determined</u> by motion of 

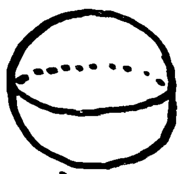
(SO(3)'s worth of rotations

(SO(2)'s worth of rotations

Def of G/H Cartan Geometry	Hamster Geometry Example
$P \downarrow M$ principal H-bundle	$P \downarrow M$ bundle of "hamster configurations" on M (position, direction) — a princ. $SO(2)$-bundle. (M: surface)
$A: TP \longrightarrow \mathfrak{g}$ a $\mathfrak{g}$-valued 1-form on P, satisfying:	$A: TP \longrightarrow \mathfrak{so}(3)$ (TP: tiny changes in hamster configuration; $\mathfrak{so}(3)$: tiny rotation of the hamsterball)
1) $A_x : T_xP \longrightarrow \mathfrak{g}$ is an isomorphism $\forall x \in M$	1) Hamster can move in exactly one way to get any desired tiny rotation of the ball (no slipping constraint!)
2) $R_h^*(A) = Ad(h^{-1})A$ $\forall h \in H$ ("H-equivariance")	2) No absolute significance to the direction the hamster is facing
3) $A(\tilde{X}) = X$ $\forall X \in \mathfrak{h}$ ($\tilde{X}$: vertical vector field corresponding to $X \in \mathfrak{h}$)	3) The hamsterball does not move when the hamster does a pure rotation (no twisting constraint!)

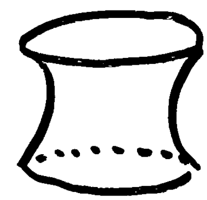
THE MORAL

Hamster Geometry		MacDowell-Mansouri
$SO(3)$	$\longrightarrow$	$SO(4,1)$
$SO(2)$	$\longrightarrow$	$SO(3,1)$



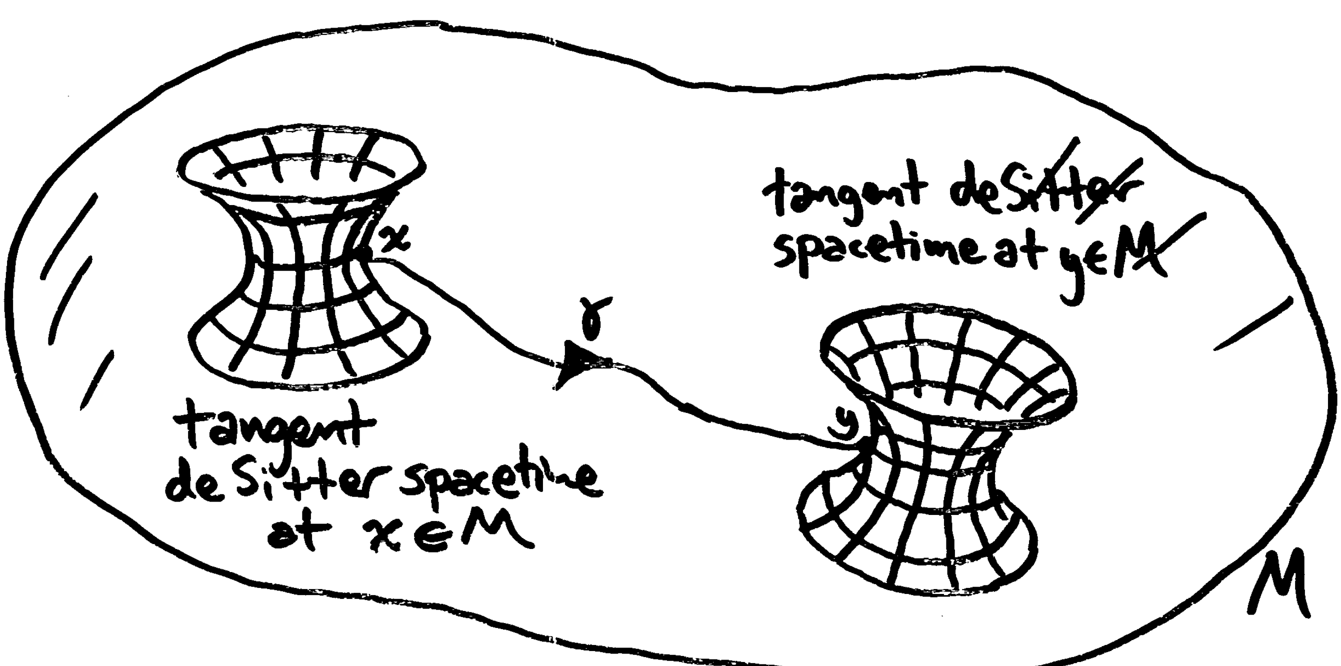
sphere

$\longrightarrow$



de Sitter spacetime

So: MacDowell-Mansouri gravity is all about rolling a model of de Sitter spacetime on physical spacetime!



For more details and references, see my paper

MacDowell–Mansouri Gravity and Cartan Geometry

gr-qc/0611154

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