SPACETIME GEOMETRY
AND
CARTAN CONNECTIONS

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MacDowell-Mansouri Gravity

MacDowell & Mansouri (PRL 38(1977)) found an action for GR with cosmological constant as an $SO(4,1)$ gauge theory:

$$S_{MM} \propto \int F^{ij} \wedge F^{kl} \varepsilon_{ijkl5}$$

$F = dA + A \wedge A$
Curv. of $SO(4,1)$ connection

How does this give GR?

$so(4,1) = so(3,1) \oplus \mathbb{R}^{3,1}$

$so(3,1)$ conn.

$A = \omega + \sqrt{\frac{4}{3}} \mathfrak{e}$
Coframefield

$\Rightarrow F = R + \frac{\Lambda}{3} \mathfrak{e} \wedge \mathfrak{e} + d_{\mathfrak{e}} \mathfrak{e}$
Curvature + $\Lambda$ term + torsion

Using this, $S_{MM} = S_{Palatini} + (\text{topological term})$
What's the geometric meaning of breaking the symmetry from $SO(4,1)$ to $SO(3,1)$?
Easier: break from $SO(3)$ to $SO(2)$.

$$so(3) = so(2) \oplus \mathbb{R}^2$$

 Tiny rotation tiny rotation fixing tiny rotation
 of a sphere chosen basepoint moving basept.

CLAIM: The $SO(3)/SO(2)$-analog of the
MacDowell-Mansouri connection is perfect for describing

 rolling a ball on a surface w/o slipping!

In fact, connections of this type were invented by Cartan in the 1920s, but they often haven't seen the appreciation they deserve!
To understand $SO(3)/SO(2)$ Cartan connections (and rolling w/o slipping) it is helpful to introduce an "observer"...

Main point:

Motion of $SO(3)$'s worth of rotations is determined by motion of $SO(2)$'s worth of rotations
Def of $G/H$ Cartan Geometry

$P \downarrow \quad \text{principal } H\text{-bundle} \quad M$

$A : TP \rightarrow \mathfrak{g}$

a $\mathfrak{g}$-valued 1-form on $P$, satisfying:

1) $A_x : T_x P \rightarrow \mathfrak{g}$
   is an isomorphism $\forall x \in M$

2) $R^*_h(A) = \text{Ad}(h^{-1})A$
   $\forall h \in H$ ("$H$-equivariance")

3) $A(\tilde{X}) = X \quad \forall X \in \mathfrak{h}$
   $\text{Vertical vector field corresponding to } X \in \mathfrak{h}$

Hamster Geometry Example

$P \downarrow \quad \text{bundle of "hamster configurations" on } M$
   $\subseteq (\text{position, direction})$
   $\rightarrow$ a princ. $\mathfrak{so}(2)$-bundle.

$M \quad \text{surface}$

$A : TP \rightarrow \mathfrak{so}(3)$

tiny changes in hamster configuration

tiny rotation of the hamsterball

1) Hamster can move in exactly one way to get an desired tiny rotation of the ball
   (no slipping constraint!)

2) No absolute significance to the direction the hamster is facing

3) The hamsterball does not move when the hamster does a pure rotation
   (no twisting constraint!)
THE MORAL

Hamster Geometry

SO(3) \rightarrow SO(4,1)
SO(2) \rightarrow SO(3,1)

sphere \rightarrow de Sitter spacetime

So: MacDowell-Mansouri gravity is all about rolling a model of de Sitter spacetime on physical spacetime!
For more details and references, see my paper

MacDowell-Mansouri Gravity
and Cartan Geometry

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