Cartan Geometry and MacDowell–Mansouri Gravity: The Work of Derek Wise

John C. Baez

Groupe de Théorie de l'Astroparticule et Cosmologie
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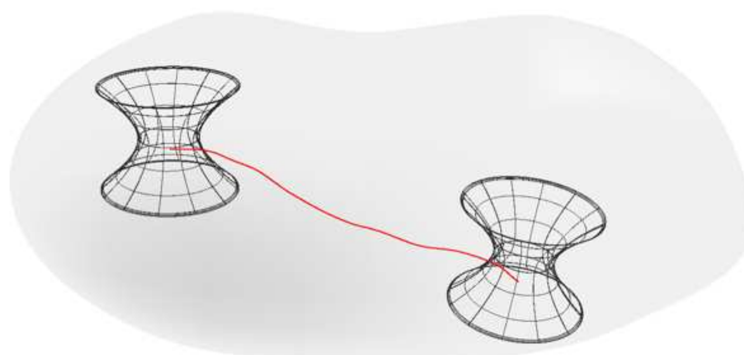
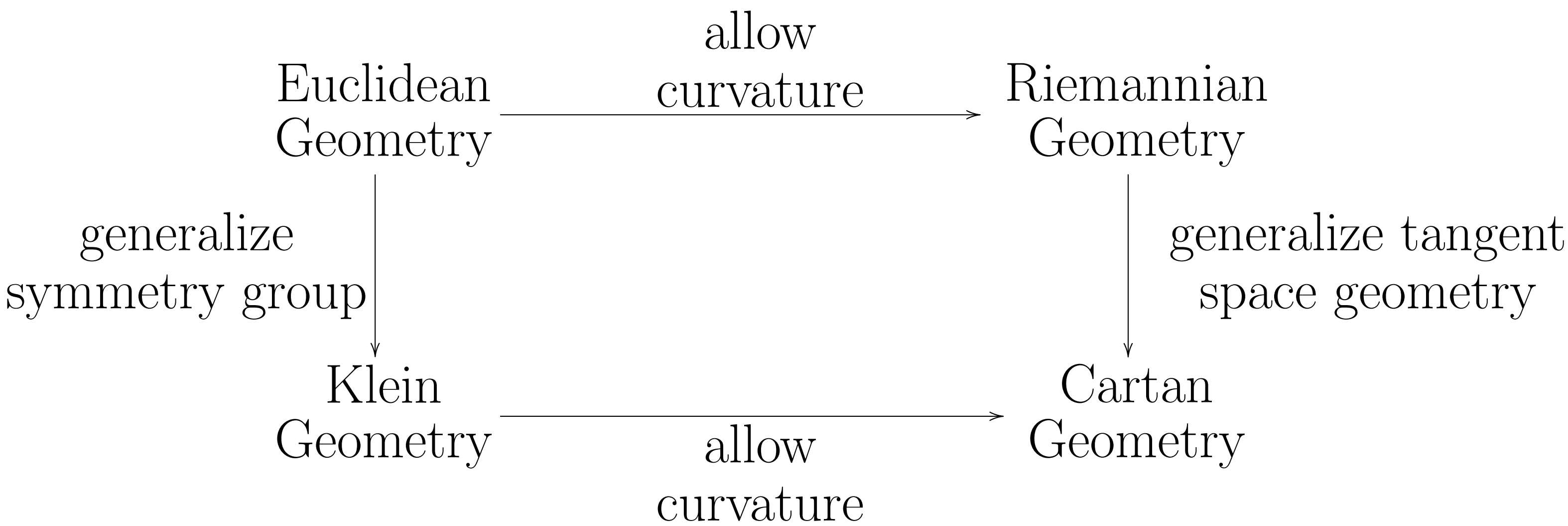


For more, see: http://math.ucr.edu/∼derek

Cartan Geometry



In Klein's approach, a geometry is a **homogeneous space**: a manifold X on which a Lie group G acts transitively, with **stabilizer** subgroup

$$H = \{g \in G : gx = x\}$$

given $x \in X$. We have

$$X = G/H$$

The first examples:

$$\begin{array}{rcll} S^2 & = & \mathrm{SO}(3)/\mathrm{SO}(2) & \text{curvature} > 0 \\ \mathbb{R}^2 & = & \mathrm{ISO}(2)/\mathrm{SO}(2) & \text{curvature} = 0 \\ H^2 & = & \mathrm{SO}(2,1)/\mathrm{SO}(2) & \text{curvature} < 0 \end{array}$$

The most important for physics:

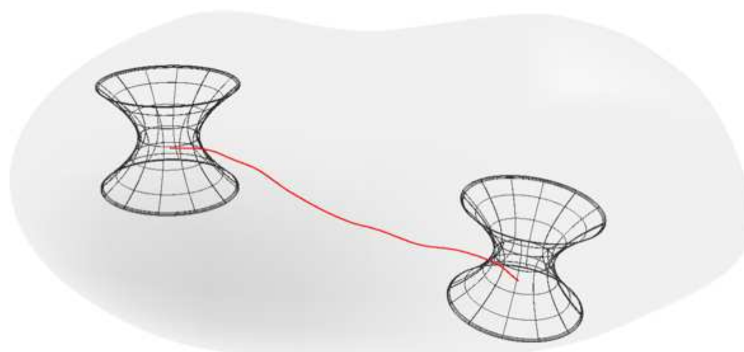
$$\begin{array}{rcll} \text{DeSitter} & = & \mathrm{SO}(4,1)/\mathrm{SO}(3,1) & \Lambda > 0 \\ \text{Minkowski} & = & \mathrm{ISO}(3,1)/\mathrm{SO}(3,1) & \Lambda = 0 \\ \text{anti-DeSitter} & = & \mathrm{SO}(3,2)/\mathrm{SO}(3,1) & \Lambda < 0 \end{array}$$

In Riemann's approach, we describe the geometry of an n-dimensional manifold M using a copy of Euclidean $\mathbb{R}^n$ at each point $p \in M$: the *tangent space*.

In Cartan's approach, we describe the geometry of M using a copy of G/H at each point: the *tangent Klein geometry*. We require

$$\dim(G/H) = \dim(M)$$

A 'Cartan connection' describes how to roll this copy of G/H along M:



Symmetric Spaces

A simplification: all the homogeneous spaces mentioned are **symmetric spaces**. This implies there is a subspace of 'infinitesimal translations'

$$\mathfrak{p} \subseteq \mathfrak{g}$$

with

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$$

and

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$$

In short, $\mathfrak{g}$ becomes a $\mathbb{Z}/2$-graded Lie algebra with:

- $\mathfrak{h}$ as 'even' part;
- $\mathfrak{p}$ as 'odd' part.

Locally, a Cartan connection can be described by a $\mathfrak{g}$-valued 1-form on M, say A. Since we're assuming G/H is a symmetric space, we have:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$$

So, we can write:

$$A = \omega + e$$

where:

- ω takes values in $\mathfrak{h}$.
- e takes values in $\mathfrak{p}$.

For us, this naïve *local* picture of Cartan geometry will suffice.

A 2-Dimensional Example

If

$$G/H = \mathrm{SO}(3)/\mathrm{SO}(2) = S^2$$

then the $\mathbb{Z}/2$-grading

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$$

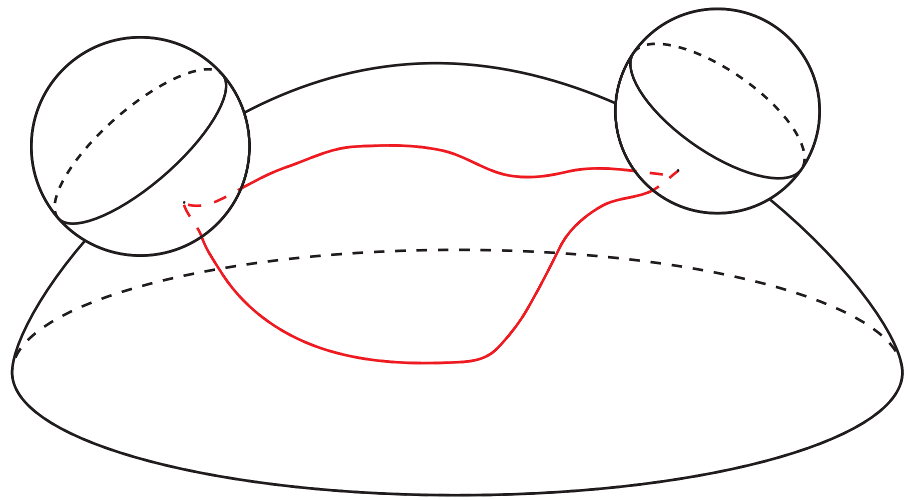
is just

$$\mathfrak{so}(3) = \mathfrak{so}(2) \oplus \mathbb{R}^2$$

since:

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & \\ -a & 0 & \\ & & \end{pmatrix} + \begin{pmatrix} & & b \\ & & c \\ -b & -c & \end{pmatrix}$$

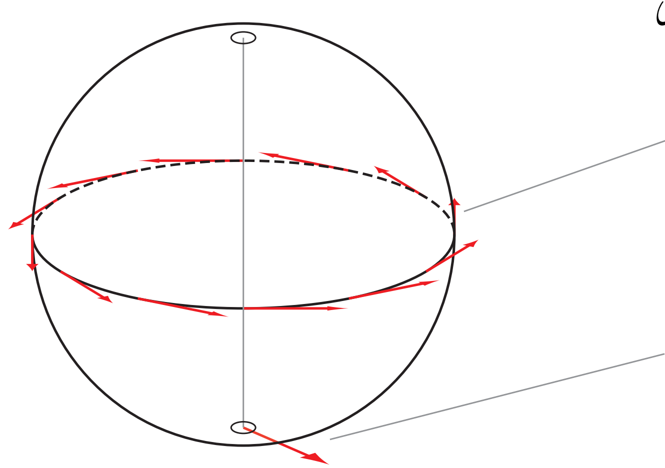
In this example, a Cartan connection A on a 2-manifold M describes how the tangent sphere rotates as we roll it along M:



Given a tangent vector v on M, $A(v) \in \mathfrak{so}(3)$ says how the tangent sphere rotates as we roll it in the direction v. Writing

$$A(v) = \omega(v) + e(v)$$

we see:



A (3+1)-Dimensional Example

If

$$G/H = \mathrm{SO}(4,1)/\mathrm{SO}(3,1)$$

is DeSitter spacetime, we have a $\mathbb{Z}/2$-grading:

$$\mathfrak{so}(4,1) = \mathfrak{so}(3,1) \oplus \mathbb{R}^{3,1}$$

If A is a Cartan connection on the 4-manifold M, $A(v) \in \mathfrak{so}(4,1)$ describes how the tangent DeSitter spacetime 'rotates' as we roll it in the direction v. Writing

$$A = \omega + e$$

we see:

- $\omega(v) \in \mathfrak{so}(3,1)$. Indeed, ω is an $\mathrm{SO}(3,1)$ connection.
- $e(v) \in \mathbb{R}^{3,1}$. Indeed, e is a coframe field, or 'cotetrad'.

Whenever G/H is a symmetric space, the **curvature** of a Cartan connection A is given by:

$$\begin{aligned} F &= dA + A \wedge A \\ &= d(\omega + e) + (\omega + e) \wedge (\omega + e) \end{aligned}$$

The curvature is the sum of a $\mathfrak{g}$-valued part, the **corrected curvature**:

$$\widehat{F} = d\omega + \omega \wedge \omega + e \wedge e$$

and a $\mathfrak{p}$-valued part, the **torsion**:

$$T = de + [\omega, e]$$

The corrected cuvature can be written as:

$$\widehat{F} = R + e \wedge e$$

where

$$R = d\omega + \omega \wedge \omega$$

is the usual curvature of ω — which in applications to gravity is the **Riemann tensor**.

A Cartan connection gives M a geometry locally the same as the Klein geometry G/H if and only if:

- $F = 0$, so the torsion T and corrected curvature $\widehat{F}$ vanish.
- The coframe field e is **nondegenerate**, meaning

$$e\colon T_x M \to \mathfrak{p}$$

  is invertible for all $x \in M$.

Generalized Chern–Simons Gravity

For any 3d symmetric space G/H and 3-manifold M with Cartan connection A, Derek Wise defines **generalized Chern–Simons gravity**, with action:

$$S = \frac{1}{\sqrt{\Lambda}} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

Here

$$\mathrm{tr} \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$$

is an invariant inner product on $\mathfrak{g}$, and

$$A = \omega + \sqrt{\Lambda}\, e$$

is any Cartan connection.

Λ is a generalization of the cosmological constant.

The equations of motion for Chern–Simons theory say:

$$F = 0$$

So, the solutions describe Cartan geometries that are locally the same as the Klein geometry G/H — at least when the coframe field e is nondegenerate.

Witten showed in the 3d DeSitter example $G/H = SO(3,1)/SO(2,1)$ that these 'locally Kleinian' geometries are just *vacuum solutions of* $(2+1)d$ *gravity with cosmological constant* $\Lambda > 0$.

The anti-DeSitter case works similarly.

In both these cases we can pick an inner product 'tr' on $\mathfrak{g}$ with $\mathfrak{h} \perp \mathfrak{h}$ and $\mathfrak{p} \perp \mathfrak{p}$. Whenever this holds, we have:

$$S = \int_M \mathrm{tr}(e \wedge R + \frac{\Lambda}{3} e \wedge e \wedge e)$$

This generalizes the Palatini action for $(2+1)$d gravity!

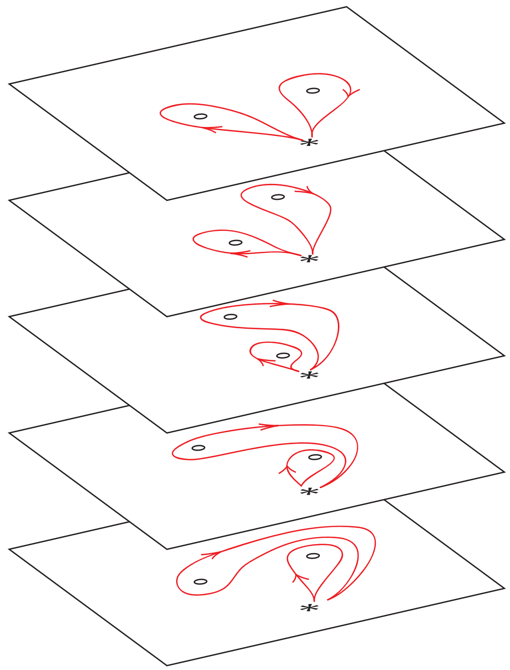
Wise then studies *point particles* in generalized Chern–Simons gravity. These arise as conical singularities in the Cartan connection.

The holonomy around a particle gives an element of G; gauge transformations conjugate this by $h \in H$. So, particles are classified by **relative conjugacy classes**

$$\{hgh^{-1} \colon \ h \in H\} \subseteq G$$

In the example $G/H = SO(3,1)/SO(2,1)$, a relative conjugacy class specifies the particle's mass and total angular momentum.

These particles obey braid group statistics:



and nonstandard rules for adding energy-momenta: 'doubly special relativity'!

Generalized MacDowell–Mansouri Gravity

For any 4d symmetric space G/H and 4-manifold M with Cartan connection A, Wise defines **generalized MacDowell–Mansouri gravity**, with action:

$$S = \frac{1}{2\Lambda} \int_M \mathrm{tr}(\widehat{F} \wedge \widehat{F})$$

Here tr is an invariant inner product on $\mathfrak{g}$, and $\widehat{F}$ is the corrected curvature of the Cartan connection A.

Recall:

$$\begin{array}{lrcl}
\text{Cartan connection:} & A & = & \omega + \sqrt{\Lambda}\, e \\
\text{curvature:} & F & = & dA + A \wedge A \;= \widehat{F} + T \\
\text{corrected curvature:} & \widehat{F} & = & R + \Lambda\, e \wedge e \\
\text{torsion:} & T & = & de + [\omega, e] \\
\text{Riemann tensor:} & R & = & d\omega + \omega \wedge \omega
\end{array}$$

Whenever $\mathfrak{h} \perp \mathfrak{p}$ the equations of motion say:

$$e \wedge T = 0, \qquad e \wedge R + \Lambda\, e \wedge e \wedge e = 0$$

MacDowell and Mansouri showed in the DeSitter example $G/H = SO(4,1)/SO(3,1)$ that these equations describe *vacuum solutions of* $(3+1)d$ *general relativity with cosmological constant* $\Lambda > 0$ — at least when the coframe field e is nondegenerate, which implies $T = 0$. Then the second equation is Einstein's equation.

The anti-DeSitter case works similarly.

Generalized MacDowell–Mansouri gravity is *not* purely topological. However, Freidel and Starodubtsev noted that MacDowell–Mansouri gravity is a *perturbation* of a topological field theory, where the coupling constant is $\Lambda \sim 10^{-120}$.

This holds for all generalized MacDowell–Mansouri gravities! To see this, consider the perturbed BF action:

$$S = \int_M \mathrm{tr}(B \wedge F - \frac{\Lambda}{2} \widehat{B} \wedge \widehat{B})$$

where F is the curvature of the Cartan connection, B is a $\mathfrak{g}$-valued 2-form, and $\widehat{B}$ is its $\mathfrak{h}$-valued part.

The equations of motion say:

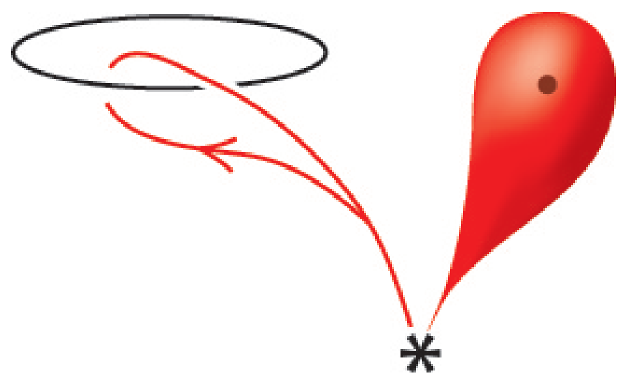
$$F = \Lambda \widehat{B}, \qquad d_A B = 0$$

When $\Lambda = 0$ these equations are 'purely topological': no gauge-invariant local degrees of freedom. But when $\Lambda \neq 0$, they imply the MacDowell–Mansouri equations!

Derek Wise then studies *particles and 1-branes* in the $\Lambda = 0$ theory, which is just a BF theory:

- 1-branes arise as conical singularities in the Cartan connection A along curves in space.
- Particles arise as singularities in the B field at points in space.

Here we use the fact that A and B form a 'flat 2-connection', which we can use to define holonomies for loops *and surfaces!*



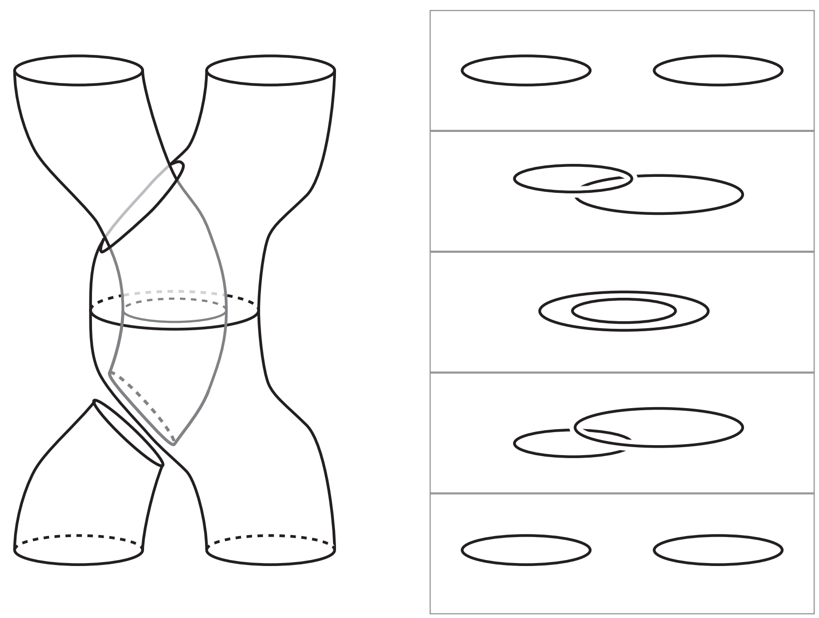
The holonomy around a 1-brane gives an element of G; gauge transformations conjugate this by $h \in H$. So, 1-branes are classified by relative conjugacy classes:

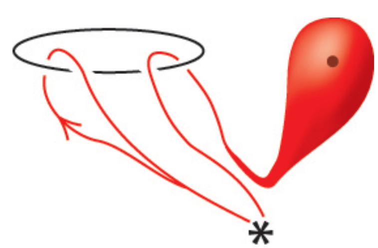
$$\{hgh^{-1} \colon \; h \in H\} \subseteq G$$

The '2-holonomy' around a particle gives an element of $\mathfrak{g}$; gauge transformations conjugate this by $h \in H$. So, particles are classified by **relative adjoint orbits**:

$$\{\mathrm{Ad}(h)(x) \colon \; h \in H\} \subseteq \mathfrak{g}$$

The particles and 1-branes obey exotic statistics, governed by the topology:





The Big Question

How much of this structure survives when we *perturb* around a Klein geometry and consider Cartan geometries — or in other words, perturb around BF theory and consider MacDowell–Mansouri gravity?