The Classifying Space of a Topological 2-Group

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Abstract

Categorifying the concept of topological group, one obtains the notion of a 'topological 2-group'. This in turn allows a theory of 'principal 2-bundles' generalizing the usual theory of principal bundles. It is well-known that under mild conditions on a topological group G and a space M, principal G-bundles over M are classified by either the Čech cohomology $\check{H}^1(M, G)$ or the set of homotopy classes $[M, BG]$, where BG is the classifying space of G. Here we review work by Bartels, Jurčo, Baas–Bökstedt–Kro, and others generalizing this result to topological 2-groups and even topological 2-categories. We explain various viewpoints on topological 2-groups and the Čech cohomology $\check{H}^1(M, \mathcal{G})$ with coefficients in a topological 2-group $\mathcal{G}$, also known as 'nonabelian cohomology'. Then we give an elementary proof that under mild conditions on M and $\mathcal{G}$ there is a bijection $\check{H}^1(M, \mathcal{G}) \cong [M, B|\mathcal{G}|]$ where $B|\mathcal{G}|$ is the classifying space of the geometric realization of the nerve of $\mathcal{G}$. Applying this result to the 'string 2-group' $\mathrm{String}(G)$ of a simply-connected compact simple Lie group G, it follows that principal $\mathrm{String}(G)$-2-bundles have rational characteristic classes coming from elements of $H^*(BG, \mathbb{Q})/\langle c\rangle$, where c is any generator of $H^4(BG, \mathbb{Q})$.

1 Introduction

Recent work in higher gauge theory has revealed the importance of categorifying the theory of bundles and considering '2-bundles', where the fiber is a topological category instead of a topological space [5]. These structures show up not only in mathematics, where they form a useful generalization of nonabelian gerbes [10], but also in physics, where they can be used to describe parallel transport of strings [31, 32].

The concepts of 'Čech cohomology' and 'classifying space' play a well-known and fundamental role in the theory of bundles. For any topological group G, principal G-bundles over a space M are classified by the first Čech cohomology

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of M with coefficients in G. Furthermore, under some mild conditions, these Čech cohomology classes are in 1-1 correspondence with homotopy classes of maps from M to the classifying space BG. This lets us define characteristic classes for bundles, coming from cohomology classes for BG.

All these concepts and results can be generalized from bundles to 2-bundles. Bartels [6] has defined principal $\mathcal{G}$-2-bundles where $\mathcal{G}$ is a 'topological 2-group': roughly speaking, a categorified version of a topological group. Furthermore, his work shows how principal $\mathcal{G}$-2-bundles over M are classified by $\check{H}^1(M, \mathcal{G})$, the first Čech cohomology of M with coefficients in $\mathcal{G}$. This form of cohomology, also known as 'nonabelian cohomology', is familiar from work on nonabelian gerbes [9, 19].

In fact, under mild conditions on $\mathcal{G}$ and M, there is a 1-1 correspondence between $\check{H}^1(M, \mathcal{G})$ and the set of homotopy classes of maps from M to a certain space $B|\mathcal{G}|$: the classifying space of the geometric realization of the nerve of $\mathcal{G}$. So, $B|\mathcal{G}|$ serves as a classifying space for the topological 2-group $\mathcal{G}$! This paper seeks to provide an introduction to topological 2-groups and nonabelian cohomology leading up to a self-contained proof of this fact.

In his pioneering work on this subject, Jurčo [22] asserted that a certain space homotopy equivalent to ours is a classifying space for the first Čech cohomology with coefficients in $\mathcal{G}$. However, there are some gaps in his argument for this assertion (see Section 5.2 for details).

Later, Baas, Bökstedt and Kro [2] gave the definitive treatment of classifying spaces for 2-bundles. For any 'good' topological 2-category $\mathcal{C}$, they construct a classifying space $B\mathcal{C}$. They then show that for any space M with the homotopy type of a CW complex, concordance classes of 'charted $\mathcal{C}$-2-bundles' correspond to homotopy classes of maps from M to $B\mathcal{C}$. In particular, a topological 2-group is just a topological 2-category with one object and with all morphisms and 2-morphisms invertible — and in this special case, their result *almost* reduces to the fact mentioned above.

There are some subtleties, however. Most importantly, while their 'charted $\mathcal{C}$-2-bundles' reduce precisely to our principal $\mathcal{G}$-2-bundles, they classify these 2-bundles up to concordance, while we classify them up to a superficially different equivalence relation. Two $\mathcal{G}$-2-bundles over a space X are 'concordant' if they are restrictions of some $\mathcal{G}$-2-bundle over $X \times [0, 1]$ to the two ends $X \times \{0\}$ and $X \times \{1\}$. This makes it easy to see that homotopic maps from X to the classifying space define concordant $\mathcal{G}$-2-bundles. We instead consider two $\mathcal{G}$-2-bundles to be equivalent if their defining Čech 1-cocycles are cohomologous. In this approach, some work is required to show that homotopic maps from X to the classifying space define equivalent $\mathcal{G}$-2-bundles. *A priori*, it is not obvious that two $\mathcal{G}$-2-bundles are equivalent in this Čech sense if and only if they are concordant. However, since the classifying space of Baas, Bökstedt and Kro is homotopy equivalent to the one we use, it follows from our work that these equivalence relations are the same — at least given $\mathcal{G}$ and M satisfying the technical conditions of both their result and ours.

We also discuss an interesting example: the 'string 2-group' String(G) of a simply-connected compact simple Lie group G [4, 20]. As its name suggests, this 2-group is of special interest in physics. Mathematically, a key fact is that $|\mathrm{String}(G)|$ — the geometric realization of the nerve of $\mathrm{String}(G)$ — is the 3-connected cover of G. Using this, one can compute the rational cohomology of $B|\mathrm{String}(G)|$. This is nice, because these cohomology classes give 'characteristic classes' for principal $\mathcal{G}$-2-bundles, and when M is a manifold one can hope to compute these in terms of a connection and its curvature, much as one does for ordinary principal bundles with a Lie group as structure group.

Section 2 is an overview, starting with a review of the classic results that people are now categorifying. Section 3 reviews four viewpoints on topological 2-groups. Section 4 explains nonabelian cohomology with coefficients in a topological 2-group. Finally, in Section 5 we prove the results stated in Section 2, and comment a bit further on the work of Jurčo and Baas–Bökstedt–Kro.

2 Overview

Once one knows about 'topological 2-groups', it is irresistibly tempting to generalize all ones favorite results about topological groups to these new entities. So, let us begin with a quick review of some classic results about topological groups and their classifying spaces.

Suppose that G is a topological group. The Čech cohomology $\check{H}^1(M,G)$ of a topological space M with coefficients in G is a set carefully designed to be in 1-1 correspondence with the set of isomorphism classes of principal G-bundles on M. Let us recall how this works.

First suppose $\mathcal{U} = \{U_i\}$ is an open cover of M and P is a principal G-bundle over M that is trivial when restricted to each open set U_i. Then by comparing local trivialisations of P over U_i and U_j we can define maps $g_{ij} \colon U_i \cap U_j \to G$: the transition functions of the bundle. On triple intersections $U_i \cap U_j \cap U_k$, these maps satisfy a cocycle condition:

$$g_{ij}(x)g_{jk}(x) = g_{ik}(x)$$

A collection of maps $g_{ij} \colon U_i \cap U_j \to G$ satisfying this condition is called a 'Čech 1-cocycle' subordinate to the cover $\mathcal{U}$. Any such 1-cocycle defines a principal G-bundle over M that is trivial over each set U_i.

Next, suppose we have two principal G-bundles over M that are trivial over each set U_i, described by Čech 1-cocycles g_{ij} and g'_{ij}, respectively. These bundles are isomorphic if and only if for some maps $f_i \colon U_i \to G$ we have

$$g_{ij}(x)f_j(x) = f_i(x)g'_{ij}(x)$$

on every double intersection $U_i \cap U_j$. In this case we say the Čech 1-cocycles are 'cohomologous'. We define $\check{H}^1(\mathcal{U},G)$ to be the quotient of the set of Čech 1-cocycles subordinate to $\mathcal{U}$ by this equivalence relation.

Recall that a 'good' cover of M is an open cover $\mathcal{U}$ for which all the non-empty finite intersections of open sets U_i in $\mathcal{U}$ are contractible. We say a space M **admits good covers** if any cover of M has a good cover that refines it. For example, any (paracompact Hausdorff) smooth manifold admits good covers, as does any simplicial complex.

If M admits good covers, $\check{H}^1(\mathcal{U}, G)$ is independent of the choice of good cover $\mathcal{U}$. So, we can denote it simply by $\check{H}^1(M, G)$. Furthermore, this set $\check{H}^1(M, G)$ is in 1-1 correspondence with the set of isomorphism classes of principal G-bundles over M. The reason is that we can always trivialize any principal G-bundle over the open sets in a good cover.

For more general spaces, we need to define the Čech cohomology more carefully. If M is a paracompact Hausdorff space, we can define it to be the limit

$$\check{H}^1(M, G) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, G)$$

over all open covers, partially ordered by refinement.

It is a classic result in topology that $\check{H}^1(M, G)$ can be understood using homotopy theory with the help of Milnor's construction [14, 28] of the classifying space BG:

Theorem 0. *Let G be a topological group. Then there is a topological space BG with the property that for any paracompact Hausdorff space M, there is a bijection*

$$\check{H}^1(M, G) \cong [M, BG]$$

Here $[X, Y]$ denotes the set of homotopy classes of maps from X into Y. The topological space BG is called the **classifying space** of G. There is a canonical principal G-bundle on BG, called the universal G-bundle, and the theorem above is usually understood as the assertion that every principal G-bundle P on M is obtained by pullback from the universal G-bundle under a certain map $M \to BG$ (the classifying map of P).

Now let us discuss how to generalize all these results to topological 2-groups. First of all, what is a '2-group'? It is like a group, but 'categorified'. While a group is a *set* equipped with *functions* describing multiplication and inverses, and an identity *element*, a 2-group is a *category* equipped with *functors* describing multiplication and inverses, and an identity *object*. Indeed, 2-groups are also known as 'categorical groups'.

A down-to-earth way to work with 2-groups involves treating them as 'crossed modules'. A crossed module consists of a pair of groups H and G, together with a homomorphism $t\colon H \to G$ and an action α of G on H satisfying two conditions, equations (4) and (5) below. Crossed modules were introduced by J. H. C. Whitehead [38] without the aid of category theory. Mac Lane and Whitehead [24] later proved that just as the fundamental group captures all the homotopy-invariant information about a connected pointed homotopy 1-type, a crossed module captures all the homotopy-invariant information about a connected pointed homotopy 2-type. By the 1960s it was clear to Verdier and others that crossed modules are essentially the same as categorical groups. The first published proof of this may be due to Brown and Spencer [12].

Just as one can define principal G-bundles over a space M for any topological group G, one can define 'principal $\mathcal{G}$-2-bundles' over M for any topological 2-group $\mathcal{G}$. Just as a principal G-bundle has a copy of G as fiber, a principal $\mathcal{G}$-2-bundle has a copy of $\mathcal{G}$ as fiber. Readers interested in more details are urged to read Bartels' thesis, available online [6]. We shall have nothing else to say about principal $\mathcal{G}$-2-bundles except that they are classified by a categorified version of Čech cohomology, denoted $\check{H}^1(M,\mathcal{G})$.

As before, we can describe this categorified Čech cohomology as a set of cocycles modulo an equivalence relation. Let $\mathcal{U}$ be a cover of M. If we think of the 2-group $\mathcal{G}$ in terms of its associated crossed module (G,H,t,α), then a cocycle subordinate to $\mathcal{U}$ consists (in part) of maps $g_{ij}\colon U_i \cap U_j \to G$ as before. However, we now 'weaken' the cocycle condition and only require that

$$t(h_{ijk})g_{ij}g_{jk} = g_{ik} \qquad (1)$$

for some maps $h_{ijk}\colon U_i \cap U_j \cap U_k \to H$. These maps are in turn required to satisfy a cocycle condition of their own on quadruple intersections, namely

$$\alpha(g_{ij})(h_{jkl})h_{ijl} = h_{ijk}h_{ikl} \qquad (2)$$

where α is the action of G on H. This mildly intimidating equation will be easier to understand when we draw it as a commuting tetrahedron — see equation (6) in Section 4. The pair (g_{ij}, h_{ijk}) is called a $\mathcal{G}$-valued **Čech 1-cocycle** subordinate to $\mathcal{U}$.

Similarly, we say two cocycles (g_{ij}, h_{ijk}) and (g'_{ij}, h'_{ijk}) are **cohomologous** if

$$t(k_{ij})g_{ij}f_j = f_i g'_{ij} \qquad (3)$$

for some maps $f_i\colon U_i \to G$ and $k_{ij}\colon U_i \cap U_j \to H$, which must make a certain prism commute — see equation (7). We define $\check{H}^1(\mathcal{U},\mathcal{G})$ to be the set of cohomology classes of $\mathcal{G}$-valued Čech 1-cocycles. To capture the entire cohomology set $\check{H}^1(M,\mathcal{G})$, we must next take a limit of the sets $\check{H}^1(\mathcal{U},\mathcal{G})$ as $\mathcal{U}$ ranges over all covers of M. For more details we refer to Section 4.

Theorem 0 generalizes nicely from topological groups to topological 2-groups. But, following the usual tradition in algebraic topology, we shall henceforth work in the category of k-spaces, i.e., compactly generated weak Hausdorff spaces. So, by 'topological space' we shall always mean a k-space, and by 'topological group' we shall mean a group object in the category of k-spaces.

Theorem 1. *Suppose that $\mathcal{G}$ is a well-pointed topological 2-group and M is a paracompact Hausdorff space admitting good covers. Then there is a bijection*

$$\check{H}^1(M,\mathcal{G}) \cong [M, B|\mathcal{G}|]$$

where the topological group $|\mathcal{G}|$ is the geometric realization of the nerve of $\mathcal{G}$.

One term here requires explanation. A topological group G is said to be 'well pointed' if $(G,1)$ is an NDR pair, or in other words if the inclusion $\{1\} \hookrightarrow G$

is a closed cofibration. We say that a topological 2-group $\mathcal{G}$ is **well pointed** if the topological groups G and H in its corresponding crossed module are well pointed. For example, any 'Lie 2-group' is well pointed: a topological 2-group is called a **Lie 2-group** if G and H are Lie groups and the maps t, α are smooth. More generally, any 'Fréchet Lie 2-group' [4] is well pointed. We explain the importance of this notion in Section 5.1.

Bartels [6] has already considered two examples of principal $\mathcal{G}$-2-bundles, corresponding to abelian gerbes and nonabelian gerbes. Let us discuss the classification of these before turning to a third, more novel example.

For an abelian gerbe [7], we first choose an abelian topological group H — in practice, usually just $\mathrm{U}(1)$. Then, we form the crossed module with $G = 1$ and this choice of H, with t and α trivial. The corresponding topological 2-group deserves to be called $H[1]$, since it is a 'shifted version' of H. Bartels shows that the classification of abelian H-gerbes matches the classification of $H[1]$-2-bundles. It is well-known that

$$|H[1]| \cong BH$$

so the classifying space for abelian H-gerbes is

$$B|H[1]| \cong B(BH)$$

In the case $H = \mathrm{U}(1)$, this classifying space is just $K(\mathbb{Z}, 3)$. So, in this case, we recover the well-known fact that abelian $\mathrm{U}(1)$-gerbes over M are classified by

$$[M, K(\mathbb{Z}, 3)] \cong H^3(M, \mathbb{Z})$$

just as principal $\mathrm{U}(1)$ bundles are classified by $H^2(M, \mathbb{Z})$.

For a nonabelian gerbe [9, 18, 19], we fix any topological group H. Then we form the crossed module with $G = Aut(H)$ and this choice of H, where $t\colon H \to G$ sends each element of H to the corresponding inner automorphism, and the action of G on H is the tautologous one. This gives a topological 2-group called $\mathrm{AUT}(H)$. Bartels shows that the classification of nonabelian H-gerbes matches the classification of $\mathrm{AUT}(H)$-2-bundles. It follows that, under suitable conditions on H, nonabelian H-gerbes are classified by homotopy classes of maps into $B|\mathrm{AUT}(H)|$.

A third application of Theorem 1 arises when G is a simply-connected compact simple Lie group. For any such group there is an isomorphism $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$ and the generator $\nu \in H^3(G, \mathbb{Z})$ transgresses to a characteristic class $c \in H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$. Associated to ν is a map $G \to K(\mathbb{Z}, 3)$ and it can be shown that the homotopy fiber of this can be given the structure of a topological group $\hat{G}$. This group $\hat{G}$ is the 3-connected cover of G. When $G = \mathrm{Spin}(n)$, this group $\hat{G}$ is known as $\mathrm{String}(n)$. In general, we might call $\hat{G}$ the **string group** of G. Note that until one picks a specific construction for the homotopy fiber, $\hat{G}$ is only defined up to homotopy — or more precisely, up to equivalence of A_∞-spaces.

In [4], under the above hypotheses on G, a topological 2-group subsequently dubbed the **string 2-group** of G was introduced. Let us denote this by

String(G). A key result about String(G) is that the topological group $|\mathrm{String}(G)|$ is equivalent to $\hat{G}$. By construction String(G) is a Fréchet Lie 2-group, hence well pointed. So, from Theorem 1 we immediately conclude:

Corollary 1. *Suppose that G is a simply-connected compact simple Lie group. Suppose M is a paracompact Hausdorff space admitting good covers. Then there are bijections between the following sets:*

- *the set of equivalence classes of principal* String(G)*-2-bundles over M,*
- *the set of isomorphism classes of principal $\hat{G}$-bundles over M,*
- $\check{H}^1(M, \mathrm{String}(G))$,
- $\check{H}^1(M, \hat{G})$,
- $[M, B\hat{G}]$.

One can describe the rational cohomology of $B\hat{G}$ in terms of the rational cohomology of BG, which is well-understood. The following result was pointed out to us by Matt Ando [1], and later discussed by Greg Ginot [17]:

Theorem 2. *Suppose that G is a simply-connected compact simple Lie group, and let $\hat{G}$ be the string group of G. Let $c \in H^4(BG, \mathbb{Q}) = \mathbb{Q}$ denote the transgression of the generator $\nu \in H^3(G, \mathbb{Q}) = \mathbb{Q}$. Then there is a ring isomorphism*

$$H^*(B\hat{G}, \mathbb{Q}) \cong H^*(BG, \mathbb{Q})/\langle c\rangle$$

where $\langle c\rangle$ is the ideal generated by c.

As a result, we obtain characteristic classes for String(G)-2-bundles:

Corollary 2. *Suppose that G is a simply-connected compact simple Lie group and M is a paracompact Hausdorff space admitting good covers. Then an equivalence class of principal* String(G)*-2-bundles over M determines a ring homomorphism*

$$H^*(BG, \mathbb{Q})/\langle c\rangle \to H^*(M, \mathbb{Q})$$

To see this, we use Corollary 1 to reinterpret an equivalence class of principal $\mathcal{G}$-2-bundles over M as a homotopy class of maps $f\colon M \to B|\mathcal{G}|$. Picking any representative f, we obtain a ring homomorphism

$$f^*\colon H^*(B|\mathcal{G}|, \mathbb{Q}) \to H^*(M, \mathbb{Q}).$$

This is independent of the choice of representative. Then, we use Theorem 2.

It is a nice problem to compute the rational characteristic classes of a principal String(G)-2-bundle over a manifold using de Rham cohomology. It should be possible to do this using the curvature of an arbitrary connection on the 2-bundle, just as for ordinary principal bundles with a Lie group as structure group. Sati, Schreiber and Stasheff [31] have recently made excellent progress on solving this problem and its generalizations to n-bundles for higher n.

3 Topological 2-Groups

In this section we recall four useful perspectives on topological 2-groups. For a more detailed account, we refer the reader to [3].

Recall that for us, a 'topological space' really means a k-space, and a 'topological group' really means a group object in the category of k-spaces. A **topological 2-group** is a groupoid in the category of topological groups. In other words, it is a groupoid $\mathcal{G}$ where the set $\mathrm{Ob}(\mathcal{G})$ of objects and the set $\mathrm{Mor}(\mathcal{G})$ of morphisms are each equipped with the structure of a topological group such that the source and target maps $s, t \colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$, the map $i \colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$ assigning each object its identity morphism, the composition map $\circ \colon \mathrm{Mor}(\mathcal{G}) \times_{\mathrm{Ob}(\mathcal{G})} \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G})$, and the map sending each morphism to its inverse are all continuous group homomorphisms.

Equivalently, we can think of a topological 2-group as a group in the category of topological groupoids. A **topological groupoid** is a groupoid $\mathcal{G}$ where $\mathrm{Ob}(\mathcal{G})$ and $\mathrm{Mor}(\mathcal{G})$ are topological spaces (or more precisely, k-spaces) and all the groupoid operations just listed are continuous maps. We say that a functor $f \colon \mathcal{G} \to \mathcal{G}'$ between topological groupoids is **continuous** if the maps $f \colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G}')$ and $f \colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G}')$ are continous. A group in the category of topological groupoids is such a thing equipped with continuous functors $m \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $\mathrm{inv} \colon \mathcal{G} \to \mathcal{G}$ and a unit object $1 \in \mathcal{G}$ satisfying the usual group axioms, written out as commutative diagrams.

This second viewpoint is useful because any topological groupoid $\mathcal{G}$ has a 'nerve' $N\mathcal{G}$, a simplicial space where the space of n-simplices consists of composable strings of morphisms

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n$$

Taking the geometric realization of this nerve, we obtain a topological space which we denote as $|\mathcal{G}|$ for short. If $\mathcal{G}$ is a topological 2-group, its nerve inherits a group structure, so that $N\mathcal{G}$ is a topological simplicial group. This in turn makes $|\mathcal{G}|$ into a topological group.

A third way to understand topological 2-groups is to view them as topological crossed modules. Recall that a **topological crossed module** (G, H, t, α) consists of topological groups G and H together with a continuous homomorphism

$$t \colon H \to G$$

and a continuous action

$$\begin{array}{rcl} \alpha \colon G \times H & \to & H \\ (g, h) & \mapsto & \alpha(g)h \end{array}$$

of G as automorphisms of H, satisfying the following two identities:

$$t(\alpha(g)(h)) = g t(h) g^{-1} \tag{4}$$

$$\alpha(t(h))(h') = h h' h^{-1}. \tag{5}$$

The first equation above implies that the map $t\colon H \to G$ is equivariant for the action of G on H defined by α and the action of G on itself by conjugation. The second equation is called the **Peiffer identity**. When no confusion is likely to result, we will sometimes denote the 2-group corresponding to a crossed module (G, H, t, α) simply by $H \to G$.

Every topological crossed module determines a topological 2-group and vice versa. Since there are some choices of convention involved in this construction, we briefly review it to fix our conventions. Given a topological crossed module (G, H, t, α), we define a topological 2-group $\mathcal{G}$ as follows. First, define the group $\mathrm{Ob}(\mathcal{G})$ of objects of $\mathcal{G}$ and the group $\mathrm{Mor}(\mathcal{G})$ of morphisms of $\mathcal{G}$ by

$$\mathrm{Ob}(\mathcal{G}) = G, \qquad \mathrm{Mor}(\mathcal{G}) = H \rtimes G$$

where the semidirect product $H \rtimes G$ is formed using the left action of G on H via α:

$$(h, g) \cdot (h', g') = (h\alpha(g)(h'), gg')$$

for $g, g' \in G$ and $h, h' \in H$. The source and target of a morphism $(h, g) \in \mathrm{Mor}(\mathcal{G})$ are defined by

$$s(h, g) = g \qquad \text{and} \qquad t(h, g) = t(h)g$$

(Denoting both the target map $t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$ and the homomorphism $t\colon H \to G$ by the same letter should not cause any problems, since the first is the restriction of the second to $H \subseteq \mathrm{Mor}(\mathcal{G})$.) The identity morphism of an object $g \in \mathrm{Ob}(G)$ is defined by

$$i(g) = (1, g).$$

Finally, the composite of the morphisms

$$\alpha = (h, g)\colon g \to t(h)g \qquad \text{and} \qquad \beta = (h', t(h)g)\colon t(h)g \to t(h'h)g'$$

is defined to be

$$\beta \circ \alpha = (h'h, g)\colon g \to t(h'h)g$$

It is easy to check that with these definitions, $\mathcal{G}$ is a 2-group. Conversely, given a topological 2-group $\mathcal{G}$, we define a crossed module (G, H, t, α) by setting G to be $\mathrm{Ob}(\mathcal{G})$, H to be $\ker(s) \subset \mathrm{Mor}(\mathcal{G})$, t to be the restriction of the target homomorphism $t\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$ to the subgroup $H \subset \mathrm{Mor}(\mathcal{G})$, and setting

$$\alpha(g)(h) = i(g)hi(g)^{-1}$$

If G is any topological group then there is a topological crossed module $1 \to G$ where t and α are trivial. The underlying groupoid of the corresponding topological 2-group has G as its space of objects, and only identity morphisms. We sometimes call this 2-group the **discrete** topological 2-group associated to G — where 'discrete' is used in the sense of category theory, not topology!

At the other extreme, if H is a topological group then it follows from the Peiffer identity that $H \to 1$ can be made into topological crossed module if and only if H is abelian, and then in a unique way. This is because a groupoid with one object and H as morphisms can be made into a 2-group precisely when H is abelian. We already mentioned this 2-group in the previous section, where we called it $H[1]$.

We will also need to talk about homomorphisms of 2-groups. We shall understand these in the strictest possible sense. So, we say a **homomorphism** of topological 2-groups is a functor such that $f\colon \mathrm{Ob}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G}')$ and $f\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G}')$ are both continuous homomorphisms of topological groups. We can also describe f in terms of the crossed modules (G, H, t, α) and (G', H', t', α') associated to $\mathcal{G}$ and $\mathcal{G}'$ respectively. In these terms the data of the functor f is described by the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{f} & H' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ G & \xrightarrow[f]{} & G' \end{array}$$

where the upper f denotes the restriction of $f\colon \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G}')$ to a map from H to H'. (We are using f to mean several different things, but this makes the notation less cluttered, and should not cause any confusion.) The maps $f\colon G \to G'$ and $f\colon H \to H'$ must both be continuous homomorphisms, and moreover must satisfy an equivariance property with respect to the actions of G on H and G' on H': we have

$$f(\alpha(g)(h)) = \alpha(f(g))(f(h))$$

for all $g \in G$ and $h \in H$.

Finally, we will need to talk about short exact sequences of topological groups and 2-groups. Here the topology is important. If G is a topological group and H is a normal topological subgroup of G, then we can define an action of H on G by right translation. In some circumstances, the projection $G \to G/H$ is a Hurewicz fibration. For instance, this is the case if G is a Lie group and H is a closed normal subgroup of G. We define a **short exact sequence** of topological groups to be a sequence

$$1 \to H \to G \to K \to 1$$

of topological groups and continuous homomorphisms such that the underlying sequence of groups is exact and the map underlying the homomorphism $G \to K$ is a Hurewicz fibration.

Similarly, we define a **short exact sequence** of topological 2-groups to be a sequence

$$1 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 1$$

of topological 2-groups and continuous homomorphisms between them such that both the resulting sequences

$$1 \to \mathrm{Ob}(\mathcal{G}') \to \mathrm{Ob}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G}'') \to 1$$

$$1 \to \mathrm{Mor}(\mathcal{G}') \to \mathrm{Mor}(\mathcal{G}) \to \mathrm{Mor}(\mathcal{G}'') \to 1$$

are short exact sequences of topological groups. Again, we can interpret this in terms of the associated crossed modules: if (G, H, t, α), (G', H', t', α') and $(G'', H'', t'', \alpha'')$ denote the associated crossed modules, then it can be shown that the sequence of topological 2-groups $1 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 1$ is exact if and only if both rows in the commutative diagram

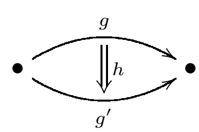
$$\begin{array}{ccccccccc} 1 & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & H'' & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G'' & \longrightarrow & 1 \end{array}$$

are short exact sequences of topological groups. In this situation we also say we have a short exact sequence of topological crossed modules.

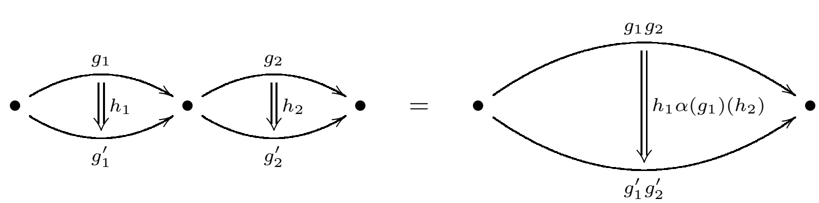
At times we shall also need a fourth viewpoint on topological 2-groups: they are strict topological 2-groupoids with a single object, say $\bullet$. In this approach, what we had been calling 'objects' are renamed 'morphisms', and what we had been calling 'morphisms' are renamed '2-morphisms'. This verbal shift can be confusing, so we will not engage in it! However, the 2-groupoid viewpoint is very handy for diagrammatic reasoning in nonabelian cohomology. We draw $g \in \mathrm{Ob}(\mathcal{G})$ as an arrow:

$$\bullet \xrightarrow{\ g\ } \bullet$$

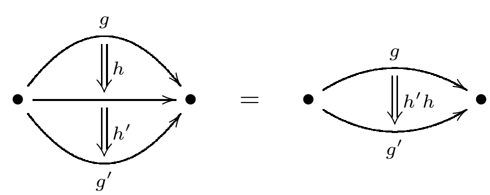
and draw $(h, g) \in \mathrm{Mor}(\mathcal{G})$ as a bigon:



where g' is the target of (h, g), namely $t(h)g$. With our conventions, horizontal composition of 2-morphisms is then given by:



while vertical composition is given by:



4 Nonabelian Cohomology

In Section 2 we gave a quick sketch of nonabelian cohomology. The subject deserves a more thorough and more conceptual explanation.

As a warmup, consider the Čech cohomology of a space M with coefficients in a topological group G. In this case, Segal [33] realized that we can reinterpret a Čech 1-cocycle as a *functor*. Suppose $\mathcal{U}$ is an open cover of M. Then there is a topological groupoid $\hat{\mathcal{U}}$ whose objects are pairs (x, i) with $x \in U_i$, and with a single morphism from (x, i) to (x, j) when $x \in U_i \cap U_j$, and none otherwise. We can also think of G as a topological groupoid with a single object $\bullet$. Segal's key observation was that a continuous functor

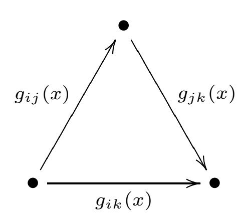
$$g: \hat{\mathcal{U}} \to G$$

is the same as a normalized Čech 1-cocycle subordinate to $\mathcal{U}$.

To see this, note that a functor $g: \hat{\mathcal{U}} \to G$ maps each object of $\hat{\mathcal{U}}$ to $\bullet$, and each morphism $(x, i) \to (x, j)$ to some $g_{ij}(x) \in G$. For the functor to preserve composition, it is necessary and sufficient to have the cocycle equation

$$g_{ij}(x) g_{jk}(x) = g_{ik}(x)$$

We can draw this suggestively as a commuting triangle in the groupoid G:

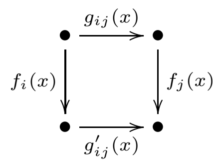


For the functor to preserve identities, it is necessary and sufficient to have the normalization condition $g_{ii}(x) = 1$.

In fact, even more is true: two cocycles g_{ij} and g'_{ij} subordinate to $\mathcal{U}$ are cohomologous if and only if the corresponding functors g and g' from $\hat{\mathcal{U}}$ to G have a continuous natural isomorphism between them. To see this, note that g_{ij} and g'_{ij} are cohomologous precisely when there are maps $f_i: U_i \to G$ satisfying

$$g_{ij}(x) f_j(x) = f_i(x) g'_{ij}(x)$$

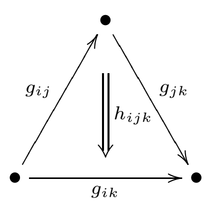
We can draw this equation as a commuting square in the groupoid G:



This is precisely the naturality square for a natural isomorphism between the functors g and g'.

One can obtain Čech cohomology with coefficients in a 2-group by categorifying Segal's ideas. Suppose $\mathcal{G}$ is a topological 2-group and let (G, H, t, α) be the corresponding topological crossed module. Now $\mathcal{G}$ is the same as a topological 2-groupoid with one object $\bullet$. So, it is no longer appropriate to consider mere *functors* from $\hat{\mathcal{U}}$ into $\mathcal{G}$. Instead, we should consider *weak 2-functors*, also known as 'pseudofunctors' [23]. For this, we should think of $\hat{\mathcal{U}}$ as a topological 2-groupoid with only identity 2-morphisms.

Let us sketch how this works. A weak 2-functor $g\colon \hat{\mathcal{U}} \to \mathcal{G}$ sends each object of $\hat{\mathcal{U}}$ to $\bullet$, and each 1-morphism $(x, i) \to (x, j)$ to some $g_{ij}(x) \in G$. However, composition of 1-morphisms is only weakly preserved. This means the above triangle will now commute only up to isomorphism:

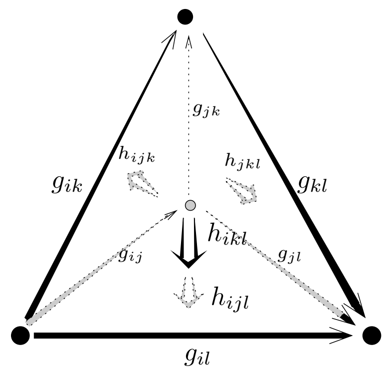


where for readability we have omitted the dependence on $x \in U_i \cap U_j \cap U_k$. Translated into equations, this triangle says that we have continuous maps $h_{ijk}\colon U_i \cap U_j \cap U_k \to H$ satisfying

$$g_{ik}(x) = t(h_{ijk}(x))g_{ij}(x)g_{jk}(x)$$

This is precisely equation (1) from Section 2.

For a weak 2-functor, it is not merely true that composition is preserved up to isomorphism: this isomorphism is also subject to a coherence law. Namely, the following tetrahedron must commute:



(6)

where again we have omitted the dependence on x. The commutativity of this tetrahedron is equivalent to the following equation:

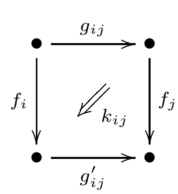
$$\alpha(g_{ij})(h_{jkl})h_{ijl} = h_{ijk}h_{ikl}$$

holding for all $x \in U_i \cap U_j \cap U_k \cap U_l$. This is equation (2).

A weak 2-functor may also preserve identity 1-morphisms only up to isomorphism. However, it turns out [6] that without loss of generality we may assume that g preserves identity 1-morphisms *strictly*. Thus we have $g_{ii}(x) = 1$ for all $x \in U_i$. We may also assume $h_{ijk}(x) = 1$ whenever two or more of the indices i, j and k are equal. Finally, just as for the case of an ordinary topological group, we require that g is a *continuous* weak 2-functor We shall not spell this out in detail; suffice it to say that the maps $g_{ij} \colon U_i \cap U_j \to G$ and $h_{ijk} \colon U_i \cap U_j \cap U_k \to H$ should be continuous. We say such continuous weak 2-functors $g \colon \hat{\mathcal{U}} \to \mathcal{G}$ are **Čech 1-cocycles** valued in $\mathcal{G}$, subordinate to the cover $\mathcal{U}$.

We now need to understand when two such cocycles should be considered equivalent. In the case of cohomology with coefficients in an ordinary topological group, we saw that two cocycles were cohomologous precisely when there was a continuous natural isomorphism between the corresponding functors. In our categorified setting we should instead use a 'weak natural isomorphism', also called a pseudonatural isomorphism [23]. So, we declare two cocycles to be **cohomologous** if there is a continuous weak natural isomorphism $f \colon g \Rightarrow g'$ between the corresponding weak 2-functors g and g'.

In a weak natural isomorphism, the usual naturality square commutes only up to isomorphism. So, $f \colon g \Rightarrow g'$ not only sends every object (x, i) of $\hat{\mathcal{U}}$ to some $f_i(x) \in G$, but also sends every morphism $(x, i) \to (x, j)$ to some $k_{ij}(x) \in H$ filling in this square:

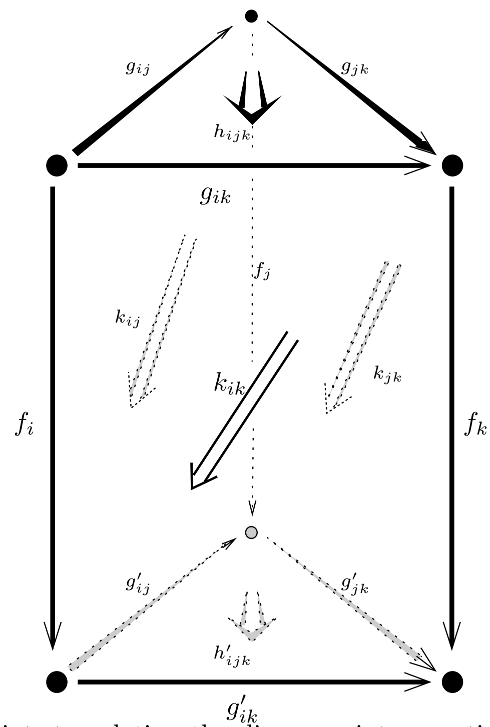


Translated into equations, this square says that

$$t(k_{ij})g_{ij}f_j = f_i g'_{ij}$$

This is equation (3).

There is also a coherence law that the k_{ij} must satisfy: they must make the following prism commute:



(7)

At this point, translating the diagrams into equations becomes tiresome and unenlightening.

It can be shown that this notion of 'cohomologousness' of Čech 1-cocycles $g\colon \hat{\mathcal{U}} \to \mathcal{G}$ is an equivalence relation. We denote by $\check{H}^1(\mathcal{U},\mathcal{G})$ the set of equivalence classes of cocycles obtained in this way. In other words, we let $\check{H}^1(\mathcal{U},\mathcal{G})$ be the set of continuous weak natural isomorphism classes of continuous weak 2-functors $g\colon \hat{\mathcal{U}} \to \mathcal{G}$.

Finally, to define $\check{H}^1(M,\mathcal{G})$, we need to take all covers into account as follows. The set of all open covers of M is a directed set, partially ordered by refinement. By restricting cocycles defined relative to $\mathcal{U}$ to any finer cover $\mathcal{V}$, we obtain a map $\check{H}^1(\mathcal{U},\mathcal{G}) \to \check{H}^1(\mathcal{V},\mathcal{G})$. This allows us to define the Čech cohomology $\check{H}^1(M,\mathcal{G})$ as a limit:

Definition 3. Given a topological space M and a topological 2-group $\mathcal{G}$, we define the **first Čech cohomology of M with coefficients in $\mathcal{G}$** to be

$$\check{H}^1(M,\mathcal{G}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U},\mathcal{G})$$

When we want to emphasize the crossed module, we will sometimes use the notation $\check{H}^1(M, H \to G)$ instead of $\check{H}^1(M, \mathcal{G})$. Note that $\check{H}^1(M, \mathcal{G})$ is a pointed set, pointed by the trivial cocycle defined relative to any open cover $\{U_i\}$ by $g_{ij} = 1$, $h_{ijk} = 1$ for all indices i, j and k.

In Theorem 1 we assume M admits good covers, so that every cover $\mathcal{U}$ of M has a refinement by a good cover $\mathcal{V}$. In other words, the directed set of good covers of M is cofinal in the set of all covers of M. As a result, in computing the limit above, it is sufficient to only consider *good* covers $\mathcal{U}$.

Finally, we remark that there is a more refined version of the set $\check{H}^1(M, \mathcal{G})$ defined using the notion of 'hypercover' [9, 11, 21]. For a paracompact space M this refined cohomology set $H^1(M, G)$ is isomorphic to the set $\check{H}^1(M, G)$ defined in terms of Čech covers. While the technology of hypercovers is certainly useful, and can simplify some proofs, our approach is sufficient for the applications we have in mind (see also the remark following the proof of Lemma 2 in subsection 5.4).

5 Proofs

5.1 Proof of Theorem 1

First, we need to distinguish between Milnor's [28] original construction of a classifying space for a topological group and a later construction introduced by Milgram, Segal and Steenrod [27, 33, 36] and further studied by May [25]. Milnor's construction is very powerful, as witnessed by the generality of Theorem 0. The later construction is conceptually more beautiful: for any topological group G, it constructs BG as the geometric realization of the nerve of the topological groupoid with one object associated to G. But, here we are performing this construction in the category of k-spaces, rather than the traditional category of topological spaces. It also seems to give a slightly weaker result: to obtain a bijection

$$\check{H}^1(M, G) \cong [M, BG]$$

all of the above cited works require some extra hypotheses on G: Segal [34] requires that G be locally contractible; May, Milgram and Steenrod require that G be well pointed. This extra hypothesis on G is required in the construction of the universal principal G-bundle EG over BG; to ensure that the bundle is locally trivial we must make one of the above assumptions on G. May's work goes further in this regard: he proves that if G is well pointed then EG is a *numerable* principal G-bundle over BG, and hence $EG \to BG$ is a Hurewicz fibration.

Another feature of this later construction is that EG comes equipped with the structure of a topological group. In the work of May and Segal, this arises from the fact that EG is the geometric realization of the nerve of a topological 2-group. We need the group structure on EG, so we will use this later construction rather than Milnor's. For further comparison of the constructions see tom Dieck [37].

We prove Theorem 1 using three lemmas that are of some interest in their own right. The second, as far as we know, is due to Larry Breen:

Lemma 1. *Let $\mathcal{G}$ be any well-pointed topological 2-group, and let (G, H, t, α) be the corresponding topological crossed module. Then:*

1. *$|\mathcal{G}|$ is a well-pointed topological group.*
2. *There is a topological 2-group $\hat{\mathcal{G}}$ such that $|\hat{\mathcal{G}}|$ fits into a short exact sequence of topological groups*
$$1 \to H \to |\hat{\mathcal{G}}| \xrightarrow{p} |\mathcal{G}| \to 1$$
3. *G acts continuously via automorphisms on the topological group EH, and there is an isomorphism $|\hat{\mathcal{G}}| \cong G \ltimes EH$. This exhibits $|\mathcal{G}|$ as $G \ltimes_H EH$, the quotient of $G \ltimes EH$ by the normal subgroup H.*

Lemma 2. *If*
$$1 \to H \xrightarrow{t} G \xrightarrow{p} K \to 1$$
is a short exact sequence of topological groups, there is a bijection
$$\check{H}^1(M, H \to G) \cong \check{H}^1(M, K)$$
Here $H \to G$ is our shorthand for the 2-group corresponding to the crossed module (G, H, t, α) where t is the inclusion of the normal subgroup H in G and α is the action of G by conjugation on H.

Lemma 3. *If*
$$1 \to \mathcal{G}_0 \xrightarrow{f} \mathcal{G}_1 \xrightarrow{p} \mathcal{G}_2 \to 1$$
is a short exact sequence of topological 2-groups, then
$$\check{H}^1(M, \mathcal{G}_0) \xrightarrow{f_*} \check{H}^1(M, \mathcal{G}_1) \xrightarrow{p_*} \check{H}^1(M, \mathcal{G}_2)$$
is an exact sequence of pointed sets.

Given these lemmas the proof of Theorem 1 goes as follows. Assume that $\mathcal{G}$ is a well-pointed topological 2-group. From Lemma 1 we see that $|\mathcal{G}|$ is a well-pointed topological group. It follows that we have a bijection
$$\check{H}^1(M, |\mathcal{G}|) \cong [M, B|\mathcal{G}|]$$
So, to prove the theorem, it suffices to construct a bijection
$$\check{H}^1(M, \mathcal{G}) \cong \check{H}^1(M, |\mathcal{G}|)$$
By Lemma 1, $|\mathcal{G}|$ fits into a short exact sequence of topological groups:
$$1 \to H \to G \ltimes EH \to |\mathcal{G}| \to 1$$

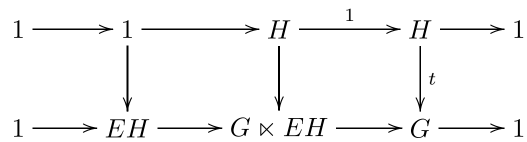
We can use Lemma 2 to conclude that there is a bijection

$$\check{H}^1(M, H \to G \ltimes EH) \cong \check{H}^1(M, |\mathcal{G}|)$$

To complete the proof it thus suffices to construct a bijection

$$\check{H}^1(M, H \to G \ltimes EH) \cong \check{H}^1(M, \mathcal{G})$$

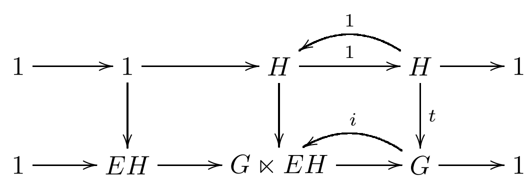
For this, observe that we have a short exact sequence of topological crossed modules:



So, by Lemma 3, we have an exact sequence of sets:

$$\check{H}^1(M, EH) \to \check{H}^1(M, H \to G \ltimes EH) \to \check{H}^1(M, H \to G)$$

Since EH is contractible and M is paracompact Hausdorff, $\check{H}^1(M, EH)$ is easily seen to be trivial, so the map $\check{H}^1(M, H \to G \ltimes EH) \to \check{H}^1(M, H \to G)$ is injective. To see that this map is surjective, note that there is a homomorphism of crossed modules going back:



where i is the natural inclusion of G in the semidirect product $G \ltimes EH$. This homomorphism going back 'splits' our exact sequence of crossed modules. It follows that $\check{H}^1(M, H \to G \ltimes EH) \to \check{H}^1(M, H \to G)$ is onto, so we have a bijection

$$\check{H}^1(M, H \to G \ltimes EH) \cong \check{H}^1(M, H \to G) = \check{H}^1(M, \mathcal{G})$$

completing the proof.

5.2 Remarks on Theorem 1

Theorem 1, asserting the existence of a classifying space for first Čech cohomology with coefficients in a topological 2-group, was originally stated in a preprint by Jurčo [22]. However, the argument given there was missing some details. In essence, the Jurčo's argument boils down to the following: he constructs a map $\check{H}^1(M, |\mathcal{G}|) \to \check{H}^1(M, \mathcal{G})$ and sketches the construction of a map $\check{H}^1(M, \mathcal{G}) \to \check{H}^1(M, |\mathcal{G}|)$. The construction of the latter map however requires some further justification: for instance, it is not obvious that one can choose a classifying map satisfying the cocycle property listed on the top of page 13 of [22]. Apart from this, it is not demonstrated that these two maps are inverses of each other.

As mentioned earlier, Jurčo and also Baas, Bökstedt and Kro [2] use a different approach to construct a classifying space for a topological 2-group $\mathcal{G}$. In their approach, $\mathcal{G}$ is regarded as a topological 2-groupoid with one object. There is a well-known nerve construction that turns any 2-groupoid (or even any 2-category) into a simplicial set [15]. Internalizing this construction, these authors turn the topological 2-groupoid $\mathcal{G}$ into a simplicial space, and then take the geometric realization of that to obtain a space. Let us denote this space by $B\mathcal{G}$. This is the classifying space used by Jurčo and Baas–Bökstedt–Kro. It should be noted that the assumption that $\mathcal{G}$ is a well-pointed 2-group ensures that the nerve of the 2-groupoid $\mathcal{G}$ is a 'good' simplicial space in the sense of Segal; this 'goodness' condition is important in the work of Baas, Bökstedt and Kro [2].

Baas, Bökstedt and Kro also consider a third way to construct a classifying space for $\mathcal{G}$. If we take the nerve $N\mathcal{G}$ of $\mathcal{G}$ we get a simplicial group, as described in Section 3 above. By thinking of each group of p-simplices $(N\mathcal{G})_p$ as a groupoid with one object, we can think of $N\mathcal{G}$ as a simplicial groupoid. From $N\mathcal{G}$ we can obtain a bisimplicial space $NN\mathcal{G}$ by applying the nerve construction to each groupoid $(N\mathcal{G})_p$. $NN\mathcal{G}$ is sometimes called the 'double nerve', since we apply the nerve construction twice. From this bisimplicial space $NN\mathcal{G}$ we can form an ordinary simplicial space $dNN\mathcal{G}$ by taking the diagonal. Taking the geometric realization of this simplicial space, we obtain a space $|dNN\mathcal{G}|$.

It turns out that this space $|dNN\mathcal{G}|$ is homeomorphic to $B|\mathcal{G}|$ [8, 30]. It can also be shown that the spaces $|dNN\mathcal{G}|$ and $B\mathcal{G}$ are homotopy equivalent — but although this fact seems well-known to experts, we have been unable to find a reference in the case of a *topological* 2-group $\mathcal{G}$. For ordinary 2-groups (without topology) the relation between all three nerves was worked out by Moerdijk and Svensson [29] and Bullejos and Cegarra [13]. In any case, since we do not use these facts in our arguments, we forgo providing the proofs here.

5.3 Proof of Lemma 1

Suppose $\mathcal{G}$ is a well-pointed topological 2-group with topological crossed module (G, H, t, α), and let $|\mathcal{G}|$ be the geometric realization of its nerve. We shall prove that there is a topological 2-group $\hat{\mathcal{G}}$ fitting into a short exact sequence of topological 2-groups

$$1 \to H \to \hat{\mathcal{G}} \to \mathcal{G} \to 1 \tag{8}$$

where H is the discrete topological 2-group associated to the topological group H. On taking nerves and then geometric realizations, this gives an exact sequence of groups:

$$1 \to H \to |\hat{\mathcal{G}}| \to |\mathcal{G}| \to 1$$

Redescribing the 2-group $\hat{\mathcal{G}}$ with the help of some work by Segal, we shall show that $|\hat{\mathcal{G}}| \cong G \ltimes EH$ and thus $|\mathcal{G}| \cong (G \ltimes EH)/H$. Then we prove that the above sequence is an exact sequence of *topological* groups: this requires checking that $|\hat{\mathcal{G}}| \to |\mathcal{G}|$ is a Hurewicz fibration. We conclude by showing that $|\hat{\mathcal{G}}|$ is well-pointed.

To build the exact sequence of 2-groups in equation (8), we construct the corresponding exact sequence of topological crossed modules. This takes the following form:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & 1 & \longrightarrow & H & \xrightarrow{1} & H & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle t} & & \\
1 & \longrightarrow & H & \xrightarrow{f} & G \ltimes H & \xrightarrow{f'} & G & \longrightarrow & 1
\end{array}$$

Here the crossed module $(G \ltimes H, H, t', \alpha')$ is defined as follows:

$$\begin{aligned} t'(h) &= (1, h) \\ \alpha'(g,h)(h') &= \alpha(t(h)g)(h') \end{aligned}$$

while f and f' are given by

$$\begin{array}{rcl} f\colon H & \to & G \ltimes H \\ h & \mapsto & (t(h), h^{-1}) \end{array}$$

$$\begin{array}{rcl} f'\colon G \ltimes H & \to & G \\ (g,h) & \mapsto & t(h)g \end{array}$$

It is easy to check that these formulas define an exact sequence of topological crossed modules. The corresponding exact sequence of topological 2-groups is

$$1 \to H \to \hat{\mathcal{G}} \to \mathcal{G} \to 1$$

where $\hat{\mathcal{G}}$ denotes the topological 2-group associated to the topological crossed module $(G \ltimes H, H, t', \alpha')$.

In more detail, the 2-group $\hat{\mathcal{G}}$ has

$$\begin{aligned}
&\mathrm{Ob}(\hat{\mathcal{G}}) = G \ltimes H \\
&\mathrm{Mor}(\hat{\mathcal{G}}) = (G \ltimes H) \ltimes H \\
&s((g,h),h') = (g,h), \qquad t((g,h),h') = (g, h'h) \\
&i(g,h) = ((g,h),1), \qquad ((g,h'h),h'') \circ ((g,h),h') = ((g,h), h''h')
\end{aligned}$$

Note that there is an isomorphism $(G \ltimes H) \ltimes H \cong G \ltimes H^2$ sending $((g,h),h')$ to $(g,(h,h'h))$. Here by $G \ltimes H^2$ we mean the semidirect product formed with the diagonal action of G on H^2, namely $g(h,h') = (\alpha(g)(h), \alpha(g)(h'))$. Thus the group $\mathrm{Mor}(\hat{\mathcal{G}})$ is isomorphic to $G \ltimes H^2$.

We can give a clearer description of the 2-group $\hat{\mathcal{G}}$ using the work of Segal [33]. Segal noted that for any topological group H, there is a 2-group $\overline{H}$ with one object for each element of H, and one morphism from any object to any other. In other words, $\overline{H}$ is the 2-group with:

$$\begin{aligned}
&\mathrm{Ob}(\overline{H}) = H \\
&\mathrm{Mor}(\overline{H}) = H^2 \\
&s(h,h') = h, \qquad t(h,h') = h' \\
&i(h) = (h,h), \qquad (h',h'') \circ (h,h') = (h,h'')
\end{aligned}$$

Moreover, Segal proved that the geometric realization $|\overline{H}|$ of the nerve of $\overline{H}$ is a model for EH. Since G acts on H by automorphisms, we can define a 'semidirect product' 2-group $G \ltimes \overline{H}$ with

$$\begin{aligned}
&\mathrm{Ob}(G \ltimes \overline{H}) = G \ltimes H \\
&\mathrm{Mor}(G \ltimes \overline{H}) = G \ltimes H^2 \\
&s(g,(h,h')) = (g,h), \qquad t(g,(h,h')) = (g,h') \\
&i(g,h) = (g,(h,h)), \qquad (g,(h',h'')) \circ (g,(h,h')) = (g,(h,h''))
\end{aligned}$$

The isomorphism $(G \ltimes H) \ltimes H \cong G \ltimes H^2$ above can then be interpreted as an isomorphism $\mathrm{Mor}(\hat{\mathcal{G}}) \cong \mathrm{Mor}(G \ltimes \overline{H})$. It is easy to check that this isomorphism is compatible with the structure maps for $\hat{\mathcal{G}}$ and $G \ltimes \overline{H}$, so we have an isomorphism of topological 2-groups:

$$\hat{\mathcal{G}} \cong G \ltimes \overline{H}$$

It follows that the nerve $N\hat{\mathcal{G}}$ of $\hat{\mathcal{G}}$ is isomorphic as a simplicial topological group to the nerve of $G \ltimes \overline{H}$. As a simplicial space it is clear that $N(G \ltimes \overline{H}) = G \times N\overline{H}$. We need to identify the simplicial group structure on $G \times N\overline{H}$.

From the definition of the products on $\mathrm{Ob}(G \ltimes \overline{H})$ and $\mathrm{Mor}(G \ltimes \overline{H})$, it is clear that the product on $N(G \ltimes \overline{H})$ is given by the simplicial map

$$(G \times N\overline{H}) \times (G \times N\overline{H}) \to G \times N\overline{H}$$

defined on p-simplices by

$$\big((g,(h_1,\ldots,h_p)),(g',(h'_1,\ldots,h'_p))\big) \mapsto (gg',(h_1\alpha(g)(h'_1),\ldots,h_p\alpha(g)(h'_p)))$$

Thus one might well call $N(G \ltimes \overline{H})$ the 'semidirect product' $G \ltimes N\overline{H}$. Since geometric realization preserves products, it follows that there is an isomorphism of topological groups

$$|\hat{\mathcal{G}}| \cong G \ltimes EH.$$

Here the semidirect product is formed using the action of G on EH induced from the action of G on H. Finally note that H is embedded as a normal subgroup of $G \ltimes EH$ through

$$\begin{array}{ccc}
H & \to & G \ltimes EH \\
h & \mapsto & (t(h), h^{-1})
\end{array}$$

It follows that the exact sequence of groups $1 \to H \to |\hat{\mathcal{G}}| \to |\mathcal{G}| \to 1$ can be identified with

$$1 \to H \to G \ltimes EH \to |\mathcal{G}| \to 1 \tag{9}$$

It follows that $|\mathcal{G}|$ is isomorphic to the quotient $G \ltimes_H EH$ of $G \ltimes EH$ by the normal subgroup H. This amounts to factoring out by the action of H on $G \ltimes EH$ given by $h(g, x) = (t(h)g, xh^{-1})$.

Next we need to show that equation (9) specifies an exact sequence of *topological* groups: in particular, that the map $G \ltimes EH \to |\mathcal{G}| = G \ltimes_H EH$ is a Hurewicz fibration. To do this, we prove that the following diagram is a pullback:

$$\begin{array}{ccc} G \ltimes EH & \longrightarrow & EH \\ \downarrow & & \downarrow \\ G \ltimes_H EH & \longrightarrow & BH \end{array}$$

Since H is well pointed, $EH \to BH$ is a numerable principal bundle (and hence a Hurewicz fibration) by the results of May [25] referred to earlier. The statement above now follows, as Hurewicz fibrations are preserved under pullbacks.

To show the above diagram is a pullback, we construct a homeomorphism

$$\alpha\colon (G \ltimes_H EH) \times_{BH} EH \to G \ltimes EH$$

whose inverse is the canonical map $\beta\colon G \ltimes EH \to (G \ltimes_H EH) \times_{BH} EH$. To do this, suppose that $([g, x], y) \in (G \ltimes_H EH) \times_{BH} EH$. Then x and y belong to the same fiber of EH over BH, so $y^{-1}x \in H$. We set

$$\alpha([g, x], y) = (t(y^{-1}x)g, y)$$

A straightforward calculation shows that α is well defined and that α and β are inverse to one another.

To conclude, we need to show that $|\mathcal{G}|$ is a well-pointed topological group. For this it is sufficient to show that $N\mathcal{G}$ is a 'proper' simplicial space in the sense of May [26] (note that we can replace his 'strong' NDR pairs with NDR pairs). For, if we follow May and denote by $F_p|\mathcal{G}|$ the image of $\coprod_{i=0}^{p} \Delta^i \times N\mathcal{G}_i$ in $|\mathcal{G}|$, it then follows from his Lemma 11.3 that $(|\mathcal{G}|, F_p|\mathcal{G}|)$ is an NDR pair for all p. In particular $(|\mathcal{G}|, F_0|\mathcal{G}|)$ is an NDR pair. Since $F_0|\mathcal{G}| = G$ and $(G, 1)$ is an NDR pair, it follows that $(|\mathcal{G}|, 1)$ is an NDR pair: that is, $|\mathcal{G}|$ is well pointed.

We still need to show that $N\mathcal{G}$ is proper. In fact it suffices to show that $N\mathcal{G}$ is a 'good' simplicial space in the sense of Segal [35], meaning that all the degeneracies $s_i\colon N\mathcal{G}_n \to N\mathcal{G}_{n+1}$ are closed cofibrations. The reason for this is that every good simplicial space is automatically proper — see the proof of Lewis' Corollary 2.4(b) [16]. To see that $N\mathcal{G}$ is good, note that every degeneracy homomorphism $s_i\colon N\mathcal{G}_n \to N\mathcal{G}_{n+1}$ is a section of the corresponding face homomorphism d_i, so $N\mathcal{G}_{n+1}$ splits as a semidirect product $N\mathcal{G}_{n+1} \cong N\mathcal{G}_n \ltimes \ker(d_i)$. Therefore, s_i is a closed cofibration provided that $\ker(d_i)$ is well pointed. But $\ker(d_i)$ is a retract of $N\mathcal{G}_{n+1}$, so $\ker(d_i)$ will be well pointed if $N\mathcal{G}_{n+1}$ is well pointed. For this, note that $N\mathcal{G}_{n+1}$ is isomorphic as a space to $G \times H^{n+1}$. Since the groups G and H are well pointed by hypothesis, it follows that $N\mathcal{G}_{n+1}$ is well pointed. Here we have used the fact that if $X \to Y$ and $X' \to Y'$ are closed cofibrations then $X \times X' \to Y \times Y'$ is a closed cofibration. □

5.4 Proof of Lemma 2

Suppose that M is a topological space admitting good covers. Also suppose that $1 \to H \xrightarrow{t} G \xrightarrow{p} K \to 1$ is an exact sequence of topological groups.

This data gives rise to a topological crossed module $H \xrightarrow{t} G$ where G acts on H by conjugation. For short we denote this by $H \to G$. The same data also gives a topological crossed module $1 \to K$. There is a homomorphism of crossed modules from $H \to G$ to $1 \to K$, arising from this commuting square:

$$\begin{array}{ccc} H & \longrightarrow & 1 \\ {\scriptstyle t}\downarrow & & \downarrow \\ G & \xrightarrow{p} & K \end{array}$$

Call this homomorphism α. It yields a map

$$\alpha_*\colon \check{H}^1(M, H \to G) \to \check{H}^1(M, 1 \to K).$$

Note that $\check{H}^1(M, 1 \to K)$ is just the ordinary Čech cohomology $\check{H}^1(M, K)$. To prove Lemma 2, we need to construct an inverse

$$\beta\colon \check{H}^1(M, K) \to \check{H}^1(M, H \to G).$$

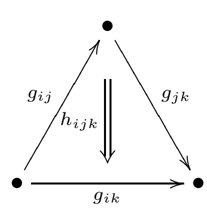
Let $\mathcal{U} = \{U_i\}$ be a good cover of M; then, as noted in Section 4 there is a bijection

$$\check{H}^1(M, K) = \check{H}^1(\mathcal{U}, K)$$

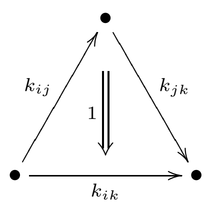
Hence to define the map β it is sufficient to define a map $\beta\colon \check{H}^1(\mathcal{U}, K) \to \check{H}^1(\mathcal{U}, H \to G)$. Let k_{ij} be a K-valued Čech 1-cocycle subordinate to $\mathcal{U}$. Then from it we construct a Čech 1-cocycle (g_{ij}, h_{ijk}) taking values in $H \to G$ as follows. Since the spaces $U_i \cap U_j$ are contractible and $p\colon G \to K$ is a Hurewicz fibration, we can lift the maps $k_{ij}\colon U_i \cap U_j \to K$ to maps $g_{ij}\colon U_i \cap U_j \to G$. The g_{ij} need not satisfy the cocycle condition for ordinary Čech cohomology, but instead we have

$$t(h_{ijk}) g_{ij} g_{jk} = g_{ik}$$

for some unique $h_{ijk}\colon U_i \cap U_j \cap U_k \to H$. In terms of diagrams, this means we have triangles



The uniqueness of h_{ijk} follows from the fact that the homomorphism $t\colon H \to G$ is injective. To show that the pair (g_{ij}, h_{ijk}) defines a Čech cocycle we need to check that the tetrahedron (6) commutes. However, this follows from the commutativity of the corresponding tetrahedron built from triangles of this form:

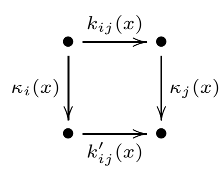


and the injectivity of t.

Let us show that this construction gives a well-defined map

$$\beta\colon \check{H}^1(M, K) = \check{H}^1(\mathcal{U}, K) \to \check{H}^1(M, H \to G)$$

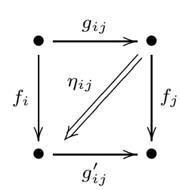
sending $[k_{ij}]$ to $[g_{ij}, h_{ijk}]$. Suppose that k'_{ij} is another K-valued Čech 1-cocycle subordinate to $\mathcal{U}$, such that k'_{ij} and k_{ij} are cohomologous. Starting from the cocycle k'_{ij} we can construct (in the same manner as above) a cocycle (g'_{ij}, h'_{ijk}) taking values in $H \to G$. Our task is to show that (g_{ij}, h_{ijk}) and (g'_{ij}, h'_{ijk}) are cohomologous. Since k_{ij} and k'_{ij} are cohomologous there exists a family of maps $\kappa_i\colon U_i \to K$ fitting into the naturality square



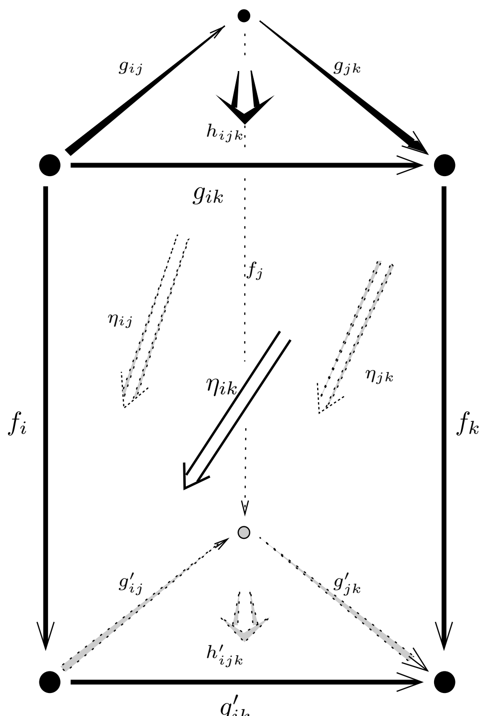
Choose lifts $f_i\colon U_i \to G$ of the various κ_i. Since $p(g_{ij}) = p(g'_{ij}) = k_{ij}$ and $p(f_i) = \kappa_i$, $p(f_j) = \kappa_j$ there is a unique map $\eta_{ij}\colon U_i \cap U_j \to H$

$$t(\eta_{ij}) g_{ij} f_j = f_i g'_{ij}.$$

So, in terms of diagrams, we have the following squares:



The triangles and squares defined so far fit together to form prisms:



It follows from the injectivity of the homomorphism t that these prisms commute, and therefore that (g_{ij}, h_{ijk}) and (g'_{ij}, h'_{ijk}) are cohomologous. Therefore we have a well-defined map $\check{H}^1(\mathcal{U}, K) \to \check{H}^1(\mathcal{U}, H \to G)$ and hence a well-defined map $\beta\colon \check{H}^1(M, K) \to \check{H}^1(M, H \to G)$.

Finally we need to check that α and β are inverse to one another. It is obvious that $\alpha \circ \beta$ is the identity on $\check{H}^1(M, K)$. To see that $\beta \circ \alpha$ is the identity on $\check{H}^1(M, H \to G)$ we argue as follows. Choose a cocycle (g_{ij}, h_{ijk}) subordinate to a good cover $\mathcal{U} = \{U_i\}$. Then under α the cocycle $[g_{ij}, h_{ijk}]$ is sent to the K-valued cocycle $[p(g_{ij})]$. But then we may take g_{ij} as our lift of $p(g_{ij})$ in the definition of $\beta(p(g_{ij}))$. It is then clear that $(\beta \circ \alpha)[g_{ij}, h_{ijk}] = [g_{ij}, h_{ijk}]$. $\square$

At this point a remark is in order. The proof of the above lemma is one place where the definition of $\check{H}^1(M, H \to G)$ in terms of hypercovers would lead to simplifications, and would allow us to replace the hypothesis that the map underlying the homomorphism $G \to K$ was a fibration with a less restrictive condition. The homomorphism of crossed modules

$$\begin{array}{ccc} H & \longrightarrow & 1 \\ {\scriptstyle t}\downarrow & & \downarrow \\ G & \xrightarrow{p} & K \end{array}$$

gives a homomorphism between the associated 2-groups and hence a simplicial map between the nerves of the associated 2-groupoids. It turns out that this simplicial map belongs to a certain class of simplicial maps with respect to which a subcategory of simplicial spaces is localized. In the formalism of hypercovers, for M paracompact, the nonabelian cohomology $\check{H}^1(M, \mathcal{G})$ with coefficients in a 2-group $\mathcal{G}$ is defined as a certain set of morphisms in this localized subcategory. It is then easy to see that the induced map $\check{H}^1(M, H \to G) \to \check{H}^1(M, K)$ is a bijection.

5.5 Proof of Lemma 3

Suppose that

$$1 \to \mathcal{G}_0 \xrightarrow{f} \mathcal{G}_1 \xrightarrow{p} \mathcal{G}_2 \to 1$$

is a short exact sequence of topological 2-groups, so that we have a short exact sequence of topological crossed modules:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H_0 & \xrightarrow{f} & H_1 & \xrightarrow{p} & H_2 & \longrightarrow & 1 \\
\downarrow & & \downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow \\
1 & \longrightarrow & G_0 & \xrightarrow{f} & G_1 & \xrightarrow{p} & G_2 & \longrightarrow & 1
\end{array}$$

Also suppose that $\mathcal{U} = \{U_i\}$ is a good cover of M, and that (g_{ij}, h_{ijk}) is a cocycle representing a class in $\check{H}^1(\mathcal{U}, \mathcal{G}_1)$. We claim that the image of

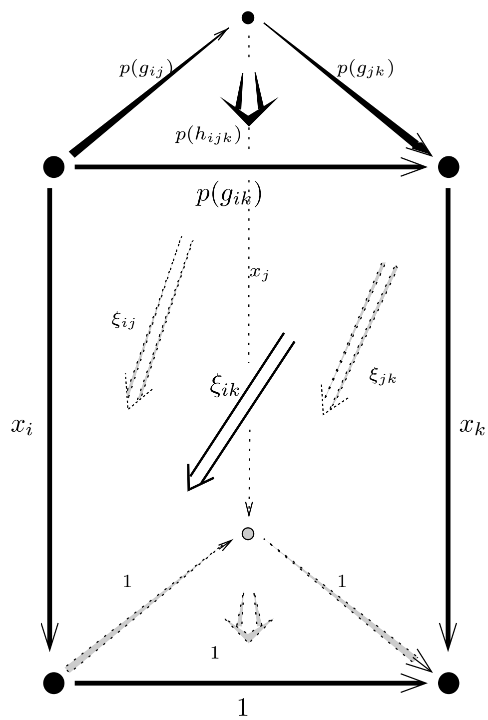
$$f_* \colon \check{H}^1(M, \mathcal{G}_0) \to \check{H}^1(M, \mathcal{G}_1)$$

equals the kernel of

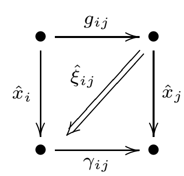
$$p_* \colon \check{H}^1(M, \mathcal{G}_1) \to \check{H}^1(M, \mathcal{G}_2).$$

If the class $[g_{ij}, h_{ijk}]$ is in the image of f_*, it is clearly in the kernel of p_*. Conversely, suppose it is in kernel of p_*. We need to show that it is in the image of f_*.

The pair $(p(g_{ij}), p(h_{ijk}))$ is cohomologous to the trivial cocycle, at least after refining the cover $\mathcal{U}$, so there exist $x_i \colon U_i \to G_2$ and $\xi_{ij} \colon U_i \cap U_j \to H_2$ such that this diagram commutes:



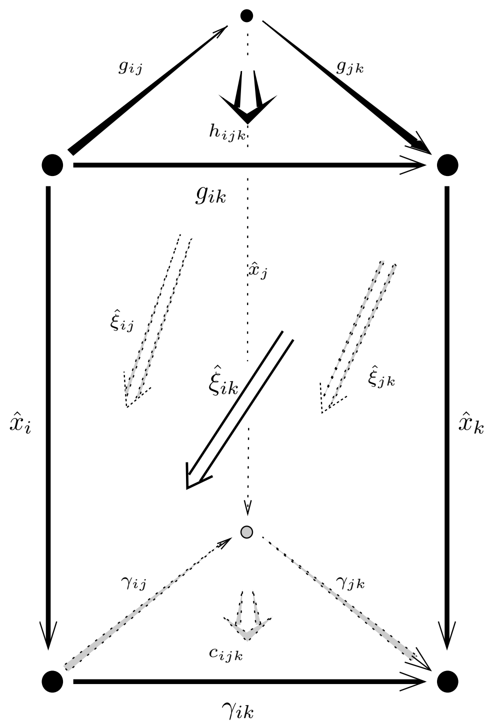
Since $p\colon G_1 \to G_2$ is a fibration and U_i is contractible, we can lift x_i to a map $\hat{x}_i\colon U_i \to G_1$. Similarly, we can lift ξ_{ij} to a map $\hat{\xi}_{ij}\colon U_i \cap U_j \to H_1$. There are then unique maps $\gamma_{ij}\colon U_i \cap U_j \to G_1$ giving squares like this:



namely

$$\gamma_{ij} = \hat{x}_i g_{ij} \hat{x}_j^{-1} t(\hat{\xi}_{ij})$$

Similarly, there are unique maps $c_{ijk}\colon U_i \cap U_j \cap U_k \to H_1$ making this prism commute:



To define c_{ijk}, we simply compose the 2-morphisms on the sides and top of the prism.

Applying p to the prism above we obtain the previous prism. So, γ_{ij} and c_{ijk} must take values in the kernel of $p\colon G_1 \to G_2$ and $p\colon H_1 \to H_2$, respectively. It follows that γ_{ij} and c_{ijk} take values in the image of f.

The above prism says that (γ_{ij}, c_{ijk}) is cohomologous to (g_{ij}, h_{ijk}), and therefore a cocycle in its own right. Since γ_{ij} and c_{ijk} take values in the image of f, they represent a class in the image of

$$f_*\colon \check{H}^1(M, \mathcal{G}_0) \to \check{H}^1(M, \mathcal{G}_1).$$

So, $[g_{ij}, h_{ijk}] = [\gamma_{ij}, c_{ijk}]$ is in the image of f_*, as was to be shown. $\square$

Proof of Theorem 2

The following proof was first described to us by Matt Ando [1], and later discussed by Greg Ginot [17].

Suppose that G is a simply-connected, compact, simple Lie group. Then the string group $\hat{G}$ of G fits into a short exact sequence of topological groups

$$1 \to K(\mathbb{Z}, 2) \to \hat{G} \to G \to 1$$

for some realization of the Eilenberg-Mac Lane space $K(\mathbb{Z}, 2)$ as a topological group. Applying the classifying space functor B to this short exact sequence gives rise to a fibration

$$K(\mathbb{Z},3) \to B\hat{G} \xrightarrow{p} BG.$$

We want to compute the rational cohomology of $B\hat{G}$.

We can use the Serre spectral sequence to compute $H^*(B\hat{G},\mathbb{Q})$. Since BG is simply connected the E_2 term of this spectral sequence is

$$E_2^{p,q} = H^p(BG,\mathbb{Q}) \otimes H^q(K(\mathbb{Z},3),\mathbb{Q}).$$

Because $K(\mathbb{Z},3)$ is rationally indistinguishable from S^3, the first nonzero differential is d_4. Furthermore, the differentials of this spectral sequence are all derivations. It follows that $d_4(y \otimes x_3) = (-1)^p y \otimes d_4(x_3)$ if $y \in H^p(BG,\mathbb{Q})$. It is not hard to identify $d_4(x_3)$ with c, the class in $H^4(BG,\mathbb{Q})$ which is the transgression of the generator ν of $H^3(G,\mathbb{Q}) = \mathbb{Q}$. It follows that the spectral sequence collapses at the E_5 stage with

$$E_5^{p,q} = E_\infty^{p,q} = \begin{cases} 0 & \text{if } q > 0 \\ H^p(BG,\mathbb{Q})/\langle c\rangle & \text{if } q = 0. \end{cases}$$

One checks that all the subcomplexes $F^i H^*(B\hat{G},\mathbb{Q})$ in the filtration of $H^*(B\hat{G},\mathbb{Q})$ are zero for $i \geq 1$. Hence $H^p(B\hat{G},\mathbb{Q}) = E_\infty^{p,0} = H^p(BG,\mathbb{Q})/\langle c\rangle$ and so Theorem 2 is proved.

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