EULER CHARACTERISTIC VERSUS HOMOTOPY CARDINALITY

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Decategorification

Sets	Categories
Elements	Objects
Equations between elements	Isomorphisms between objects
Functions	Functors
Equations between functors	Natural isomorphisms between functors

Categorification

Decategorification takes a category & produces the set of isomorphism classes of objects;

Categorification is our attempt to undo this!

What if we categorify all of mathematics?

Let's start with $\mathbb{N}$.

The obvious categorification of $\mathbb{N}$ is FinSet, the category of finite sets & functions between them:

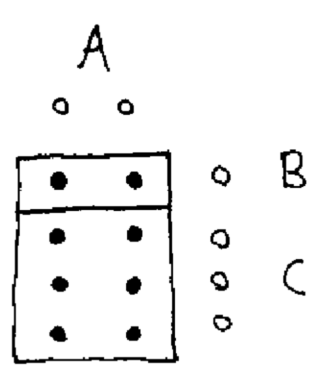
Decategorification

$\mathbb{N}$	FinSet
$+$	$+$: coproduct, aka disjoint union
$\times$	$\times$: product, aka Cartesian product
0	0 : initial object, aka empty set
1	1 : terminal object, aka 1-element set
$n = 1 + \cdots + 1$	$n = 1 + \cdots + 1$, aka n-element set

Categorification

The laws of arithmetic become natural isomorphisms:

$$A \times (B+C) \cong A \times B + A \times C$$



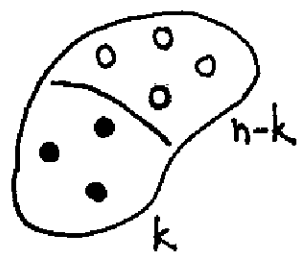
Some results in arithmetic are easy to categorify:

$$\binom{n}{k} \cong \frac{n!}{k! \times (n-k)!}$$

set of k-element subsets of n

group of permutations of n mod stabilizer of a k-element subset

$$n! := \mathrm{Aut}(n)$$



Others are surprisingly hard...

DIVISION BY 3:

Given sets A, B and isomorphism

$$f: 3 \times A \xrightarrow{\sim} 3 \times B,$$

construct isomorphism

$$g: A \xrightarrow{\sim} B.$$

1901: Bernstein claimed he could do it – never said how.

1927: Lindenbaum & Tarski claimed they could do it – never said how.

1949: Tarski claimed he'd forgotten how Lindenbaum did it – invented new method.

1989: Conway & Doyle explain how to do it: see http://math.ucr.edu/home/baez/week147.html

Try it! As a warmup, try division by 2 – much easier.

WHAT ABOUT SUBTRACTION?

Can we categorify $\mathbb{Z}$?

Schanuel noticed an obstruction to the existence of "negative sets":

If $A + B \cong 0$ in some category, then $A \cong B \cong 0$.

Nonetheless he proposed the Euler characteristic as a generalization of cardinality allowing negative integer values:

$$\chi(X) = \dim H^0(X, \mathbb{Q}) - \dim H^1(X, \mathbb{Q}) + \cdots$$

It's defined on finite CW complexes, homotopy invariant, & gets along with $+$ & $\times$.

But Schanuel preferred another variant....

EULER MEASURE

(Hadwiger, ..., Schanuel)

Let Poly be the algebra of <u>polyhedral subsets</u> of $\mathbb{R}^n$, generated by half-spaces $\{ \ell(x) \geq c \}$ via finite unions, intersections & complements. There's a unique function

$$\chi : \text{Poly} \to \mathbb{Z}$$

such that

1) $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$
2) $\chi(A) = (-1)^k$ if A homeomorphic to $\mathbb{R}^k$

Here $\mathbb{R}$ or $(0,1)$ plays the role of "the set with -1 elements". χ is invariant under homeomorphism but not homotopy, & agrees with usual Euler characteristic on <u>compact</u> sets.

EXAMPLES:

$$\chi(\bullet) = 1 \qquad \text{since } \bullet \cong \mathbb{R}^0$$

$$\chi(\circ\!-\!\circ) = -1 \qquad \text{since } \circ\!-\!\circ \cong \mathbb{R}^1$$

$$\chi(\bullet\!-\!\circ) = \chi(\bullet) + \chi(\circ\!-\!\circ)$$

$$= 1 + -1 = 0$$

$$\chi(\bullet\!-\!\bullet) = \chi(\bullet\!-\!\circ) + \chi(\bullet)$$

$$= 0 + 1 = 1$$

$$\chi(\triangle) = \chi(\bullet\!/\!\circ) + \chi(\bullet\!\setminus\!\circ) + \chi(\circ\!-\!\bullet)$$

$$= 0 + 0 + 0$$

$$= 0$$

So: Euler characteristic can be computed by "chopping up & adding up"!

Schanuel made a category Poly whose objects are polyhedral sets (in any $\mathbb{R}^n$) & whose morphisms are functions whose graphs are polyhedral sets, & proposed this as a categorification of $\mathbb{Z}$:

$$\chi(A+B) = \chi(A) + \chi(B)$$

$$\chi(A+_C B) = \chi(A) + \chi(B) - \chi(C)$$

given pushout

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, hook] & B \arrow[d] \\ A \arrow[r] & A+_C B \end{tikzcd}$$

$$\chi(A\times B) = \chi(A)\times \chi(B)$$

Propp proved many things about this, e.g. categorifying equations like $\binom{-2}{3} = -4$.

WHAT ABOUT DIVISION?

Can we categorify $\mathbb{Q}^+$?

Division is easier to understand than subtraction!

$4/2 = 2$

Action of $\mathbb{Z}_2$ on 4 has 2 orbits

$5/2 = 2\frac{1}{2}$

Action of $\mathbb{Z}_2$ on 5 has $2\frac{1}{2}$ orbits?!

Given a group G acting on a set S, let the <u>weak quotient</u> $S//G$ be the groupoid with S as objects & a morphism $g: s \to s'$ whenever $g(s) = s'$. Define the <u>cardinality</u> of a groupoid X by

$$|X| = \sum_{\substack{\text{iso classes} \\ [x] \text{ of objects}}} \frac{1}{|\mathrm{Aut}(x)|}$$

Then:

$$|S//G| = |S| / |G|$$

Carrying on in this vein, we can define the cardinality of an <u>n-groupoid</u>: a gadget with objects

$$\bullet$$

morphisms

$$\bullet \longrightarrow \bullet$$

2-morphisms

$$\begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U}] \arrow[r, bend right, ""'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

and so on, composable in various ways & all invertible (at least up to equivalence). This sounds complicated, but n-groupoids are just another way of talking about <u>homotopy n-types</u>: the nerve of an n-groupoid is a simplicial set with $\pi_i = 0$ for $i > n$. So, let's define a cardinality for homotopy types!

HOMOTOPY CARDINALITY

Baez / Dolan

Define the homotopy cardinality of a space X to be

$$|X| = \frac{1}{|\pi_1(X)|} \cdot |\pi_2(X)| \cdot \frac{1}{|\pi_3(X)|} \cdots$$

if X is connected &

$$\left| \sum_i X_i \right| = \sum_i |X_i|$$

more generally. It's defined on FinTop, the category of spaces w. finite homotopy groups, finitely many nonzero. It has:

$$|A+B| = |A| + |B|$$

$$|A \times B| = |A| \times |B|$$

$$|A \times_C B| = \frac{|A| \times |B|}{|C|}$$ given homotopy pullback

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d] & B \arrow[d, "\text{fibration}"] \\ A \arrow[r, "\text{fibration}"'] & C \end{tikzcd}$$

C ← connected

Given a fibration

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array} \leftarrow \text{connected}$$

we have $|E| = |F| \times |B|$.

<u>Looping</u> is like <u>reciprocal</u> of a connected space:

$$|\Omega X| = \frac{1}{|X|}$$

while <u>classifying space</u> is like reciprocal of a topological group:

$$|BG| = \frac{1}{|G|}$$

More general, the homotopy quotient

$$X /\!/ G := X \times_G EG$$

satisfies

$$|X /\!/ G| = |X| / |G|$$

CAN WE UNIFY EULER χ AND HOMOTOPY $|\;|$?

$$\begin{tikzcd} \mathbb{N} \arrow[r, hook] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Q}^+ \arrow[r, hook] & \mathbb{Q} \end{tikzcd} \quad \text{categorifies to} \quad \begin{tikzcd} \mathrm{FinSet} \arrow[r, hook] \arrow[d] & \mathrm{FinCW}? \arrow[d] \\ \mathrm{FinTop} \arrow[r, hook] & ??? \end{tikzcd}$$

χ and $|\;|$ are two faces of same concept, but rarely seen together unless you <u>stretch</u> them:

Example: If G is a finite group, $|BG| = \frac{1}{|G|}$.

What's $\chi(G)$? Count cells in simplicial construction:

$$\chi(G) = 1 - (|G|-1) + (|G|-1)^2 - \cdots$$

$\bullet$ — 0-cell

$\circ \xrightarrow{g} \circ$ — nondeg. 1-cells

nondeg. 2-cells

$$\stackrel{?}{=} \frac{1}{1+(|G|-1)}$$

$$= \frac{1}{|G|} \;!$$

Example: If X is a compact surface of genus g, $\chi(X) = 2-2g$.
What's $|X|$? Only $\pi_1(X)$ is nontrivial, so $|X| = \frac{1}{|\pi_1(X)|}$.

To compute $|\pi_1(X)|$, take the usual presentation of $\pi_1(X)$ and let a_n be the number of elements of length n:

$$|\pi_1(X)| = \sum_{n \geq 0} a_n$$

$$\overset{?}{=} \lim_{t \uparrow 1} \sum_{n \geq 0} a_n t^n \qquad \text{(Abel summation)}$$

$$= \frac{1}{2-2g} \; !$$

Shown by Floyd/Plotnick & Grigorchuk.

MORAL: χ AND $|\ |$ RELATED BY ANALYTIC CONTINUATION!