Higher Gauge Theory
and the String Group

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For more see:
http://math.ucr.edu/home/baez/btm22/
Categorification

sets $\leadsto$ categories
functions $\leadsto$ functors
equations $\leadsto$ natural isomorphisms

Categorification ‘boosts the dimension’ by one:

In strict categorification we keep equations as
equations. This is evil... but today we’ll do it whenever
it doesn’t cause trouble, just to save time.
Higher Gauge Theory

\[
\begin{align*}
groups & \leadsto 2\text{-groups} \\
\text{Lie algebras} & \leadsto \text{Lie 2-algebras} \\
bundles & \leadsto 2\text{-bundles} \\
connections & \leadsto 2\text{-connections}
\end{align*}
\]

Connections describe parallel transport for particles. 2-Connections describe parallel transport for strings!

We should even go beyond \( n = 2 \)… but not today.
Fix a simply-connected compact simple Lie group $G$. Then:

- The Lie algebra $\mathfrak{g}$ gives a 1-parameter family of Lie 2-algebras $\text{string}_k(\mathfrak{g})$.
- When $k \in \mathbb{Z}$, $\text{string}_k(\mathfrak{g})$ comes from a Lie 2-group $\text{String}_k(G)$.
- The ‘geometric realization of the nerve’ of $\text{String}_k(G)$ is a topological group, $|\text{String}_k(G)|$.
- Principal $\text{String}_k(G)$-2-bundles are the same as $|\text{String}_k(G)|$-bundles.
- For $k = 1$, $|\text{String}_k(G)|$ is $G$ with its 3rd homotopy group made trivial.
- We can define connections and characteristic classes for $\text{String}_k(G)$-2-bundles!
**2-Groups**

A **strict 2-group** is a category in Grp: a category with a group of objects and a group of morphisms, such that all the category operations are group homomorphisms.

The objects in a 2-group look like this:

\[ g \]

The morphisms look like this:

\[ \begin{tikzcd} \bullet & \bullet \\ \bullet & \bullet \end{tikzcd} \]

\[ g \]

\[ f \]

\[ g', g'' \]
We can multiply objects:

multiply morphisms:

and compose morphisms:
All 3 operations have a unit and inverses. All 3 are associative, so these are well-defined:

Finally, the **interchange law** holds, meaning

is well-defined.
Mac Lane and Whitehead first introduced 2-groups in the disguise of ‘crossed modules’:

\[
G_0 \xleftarrow{\partial} G_1
\]

Here \(G_0\) and \(G_1\) are groups, and \(G_0\) acts on \(G_1\) in a manner compatible with the differential \(\partial\).

To get a crossed module from a 2-group, just let \(G_0\) be the group of objects:

\[
\bullet \xrightarrow{g} \bullet
\]

and \(G_1\) be the group of morphisms starting at 1. The differential \(\partial\) is defined as follows:

\[
\bullet \xleftarrow{1} \bullet
\]

\[
\bullet \xrightarrow{f} \bullet
\]

\[
\partial(f)
\]
Lie 2-Algebras

A strict Lie 2-algebra is a category in LieAlg: a category with a Lie algebra of objects and a Lie algebra of morphisms, such that all the category operations are Lie algebra homomorphisms.

A strict Lie 2-algebra can be viewed as an ‘infinitesimal crossed module’:

\[ \mathfrak{g}_0 \xleftarrow{\partial} \mathfrak{g}_1 \]

Here \( \mathfrak{g}_0 \) and \( \mathfrak{g}_1 \) are Lie algebras, and \( \mathfrak{g}_0 \) acts as derivations of \( \mathfrak{g}_1 \) in a manner compatible with the differential \( \partial \).
**Theorem** (Mac Lane, Sinh). A 2-group is determined up to equivalence by:

- the group $G$ of isomorphism classes of objects,
- the abelian group $A$ of endomorphisms of any object,
- an action of $G$ on $A$,
- an element of $H^3(G, A)$.

**Theorem** (Gerstenhaber, Crans). A Lie 2-algebra is determined up to equivalence by:

- the Lie algebra $g$ of isomorphism classes of objects,
- the vector space $a$ of endomorphisms of any object,
- a representation of $g$ on $a$,
- an element of $H^3(g, a)$. 
Suppose $G$ is a simply-connected compact simple Lie group. Let $\mathfrak{g}$ be its Lie algebra. A lemma of Whitehead says:

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

So:

**Corollary.** For any $k \in \mathbb{R}$ there is a Lie 2-algebra $\text{string}_k(\mathfrak{g})$ for which:

- $\mathfrak{g}$ is the Lie algebra of isomorphism classes of objects;
- $\mathbb{R}$ is the vector space of endomorphisms of any object.

Every Lie 2-algebra with these properties is equivalent to $\text{string}_k(\mathfrak{g})$ for some unique $k \in \mathbb{R}$. 
Theorem. For any $k \in \mathbb{Z}$, $\text{string}_k(g)$ is the Lie 2-algebra of an infinite-dimensional Lie 2-group $\text{String}_k(G)$.

An object of $\text{String}_k(G)$ is a smooth path

$$f : [0, 2\pi] \to G$$

starting at the identity. A morphism from $f_1$ to $f_2$ is an equivalence class of pairs $(D, \alpha)$ where $D$ is a disk going from $f_1$ to $f_2$ and $\alpha \in U(1)$:
Any two such pairs \((D_1, \alpha_1)\) and \((D_2, \alpha_2)\) have a 3-ball \(B\) whose boundary is \(D_1 \cup D_2\). The pairs are equivalent when

\[
\exp \left( 2\pi ik \int_B \nu \right) = \alpha_2 / \alpha_1
\]

where \(\nu\) is the left-invariant closed 3-form on \(G\) with

\[
\nu(x, y, z) = \langle [x, y], z \rangle
\]

and \(\langle \cdot, \cdot \rangle\) is the smallest invariant inner product on \(g\) such that \(\nu\) gives an integral cohomology class.

**Theorem.** The morphisms in \(\text{String}_k(G)\) starting at the constant path form the level-\(k\) central extension of the loop group \(\Omega G\):

\[
1 \longrightarrow \text{U}(1) \longrightarrow \widehat{\Omega_k G} \longrightarrow \Omega G \longrightarrow 1
\]
For any category $\mathcal{C}$ there is a space $|\mathcal{C}|$, the **geometric realization of the nerve** of $\mathcal{C}$, built from a vertex for each object:

- A vertex for each object: $\bullet x$

an edge for each morphism:

- An edge: $\bullet \xrightarrow{f} \bullet$

a triangle for each composable pair of morphisms:

- A triangle: $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$

a tetrahedron for each composable triple:

- A tetrahedron: $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$

and so on...
A 2-group is a category with a product and inverses. So, if $G$ is a 2-group, $|G|$ is a topological group.

More generally, we can define a topological group $|G|$ for any topological 2-group $G$.

**Theorem.** For any $k \in \mathbb{Z}$, there is a short exact sequence of topological groups

$$
1 \rightarrow K(\mathbb{Z}, 2) \rightarrow |\text{String}_k(G)| \xrightarrow{p} G \xrightarrow{p} 1
$$

where $p$ is a fibration. Using this we can show:

- $\pi_1(|\text{String}_k(G)|) = 0$
- $\pi_2(|\text{String}_k(G)|) = \mathbb{Z}/k\mathbb{Z}$
- $\pi_3(|\text{String}_k(G)|) = 0$ if $k \neq 0$
**Theorem.** When $k = 1$, $|\text{String}_k(G)|$ is the ‘3-connected cover’ of $G$: the topological group formed by making the 3rd homotopy group of $G$ trivial.

For example, start with $O(n)$:

- Making $\pi_0$ trivial gives $\text{SO}(n)$.
- Making $\pi_1$ trivial gives $\text{Spin}(n)$.
- $\pi_2$ of $\text{Spin}(n)$ is already trivial.
- Making $\pi_3$ trivial gives $\text{String}(n)$.

We are claiming

$$\text{String}(n) \simeq |\text{String}_k(G)|$$

where $G = \text{Spin}(n)$ and $k = 1$. 
2-Bundles — Quick and Dirty

For any topological 2-group $\mathcal{G}$ and any space $X$, we can define a principal $\mathcal{G}$-2-bundle over $X$ to consist of:

- an open cover $U_i$ of $X$,
- continuous maps $g_{ij}: U_i \cap U_j \rightarrow \text{Ob}(\mathcal{G})$ satisfying $g_{ii} = 1$, and
- continuous maps $h_{ijk}: U_i \cap U_j \cap U_k \rightarrow \text{Mor}(\mathcal{G})$ with $h_{ijk}(x): g_{ij}(x)g_{jk}(x) \rightarrow g_{ik}(x)$
satisfying the nonabelian 2-cocycle condition:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
g_{ij} \\
g_{ik} \\
g_{il}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
g_{jk} \\
h_{ijk} \\
g_{kl}
\end{array}
\end{array}
\end{array}
\end{array} = 
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
g_{ij} \\
g_{jl} \\
g_{il}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
g_{jk} \\
h_{jkl} \\
g_{kl}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

on any quadruple intersection \( U_i \cap U_j \cap U_k \cap U_\ell \).
There’s a natural notion of ‘equivalence’ for 2-bundles over $X$, since they form a 2-category.

**Theorem.** For any topological 2-group $\mathcal{G}$ and paracompact Hausdorff space $X$, there is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over $X$,
- isomorphism classes of principal $|\mathcal{G}|$-bundles over $X$,
- homotopy classes of maps $f: X \to B|\mathcal{G}|$.

So, $B|\mathcal{G}|$ is the classifying space for $\mathcal{G}$-2-bundles.
We have homomorphisms

\[
\text{String}(n) \longrightarrow \text{Spin}(n) \longrightarrow \text{SO}(n) \longrightarrow \text{O}(n)
\]

Given an \(n\)-dimensional Riemannian manifold \(X\), we can reduce the structure group of the frame bundle from \(\text{O}(n)\) to:

- \(\text{SO}(n)\) if we have an orientation on \(X\),
- \(\text{Spin}(n)\) if we have a spin structure on \(X\),
- \(\text{String}(n)\) if we have a string structure on \(X\).

**Corollary.** For any Riemannian \(n\)-manifold \(X\), a string structure on \(X\) gives a \(\mathcal{G}\)-2-bundle over \(X\), where \(\mathcal{G} = \text{String}_k(G)\) with \(G = \text{Spin}(n)\) and \(k = 1\).
Let $\mathcal{G}$ be a Lie 2-group, $P$ the trivial principal $\mathcal{G}$-2-bundle over some smooth manifold $X$. A **2-connection** on $P$ assigns holonomies to paths in $X$:

$$\text{hol}: x \overset{\gamma}{\longrightarrow} y \quad \mapsto \quad \bullet \overset{\text{hol}(\gamma)}{\longrightarrow} \bullet \in \text{Ob}(\mathcal{G})$$

and surfaces going between paths:

$$\text{hol}: x \overset{\gamma}{\underset{\eta}{\frown}} y \quad \mapsto \quad \bullet \overset{\text{hol}(\gamma) \text{hol}(\eta)}{\underset{\text{hol}(\Sigma)}{\frown}} \bullet \in \text{Mor}(\mathcal{G})$$

in a manner preserving all 3 forms of composition:
**Theorem.** Let

\[ \mathfrak{g}_0 \xrightarrow{\partial} \mathfrak{g}_1 \]

be the infinitesimal crossed module obtained by differentiating the crossed module

\[ G_0 \xrightarrow{\partial} G_1 \]

corresponding to \( \mathcal{G} \). Then there is a 1-1 correspondence between 2-connections on \( P \to X \) and **connections**:

- a \( \mathfrak{g}_0 \)-valued 1-form \( A \) on \( X \)
- a \( \mathfrak{g}_1 \)-valued 2-form \( B \) on \( X \)

satisfying the **fake flatness** condition:

\[ dA + \frac{1}{2}[A, A] + \partial B = 0 \]
All this generalizes to nontrivial 2-bundles.

**Nice Problem.** When $\mathcal{G} = \text{String}_k(G)$, compute the real characteristic classes of a $\mathcal{G}$-2-bundle in terms of an arbitrary connection on this 2-bundle.

The homomorphism $|\mathcal{G}| \to G$ gives an algebra homomorphism:

$$ H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|\mathcal{G}|, \mathbb{R}) $$

When $k = 1$ this is onto, with kernel generated by the ‘second Chern class’ $c_2 \in H^4(BG, \mathbb{R})$.

In this case, the real characteristic classes of $\mathcal{G}$-2-bundles are just like those of $G$-bundles, but with the second Chern class killed!