Higher Gauge Theory – I

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joint work with:
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Barrett Lectures
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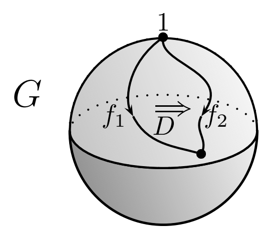
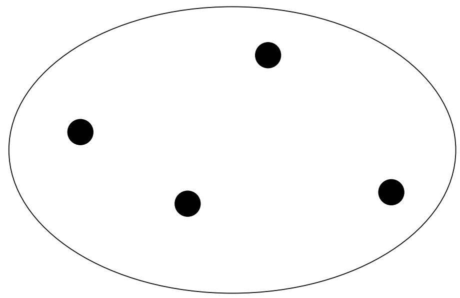


Notes and references at:

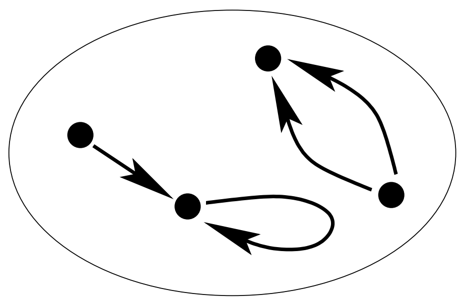
http://math.ucr.edu/home/baez/barrett/

The Big Idea

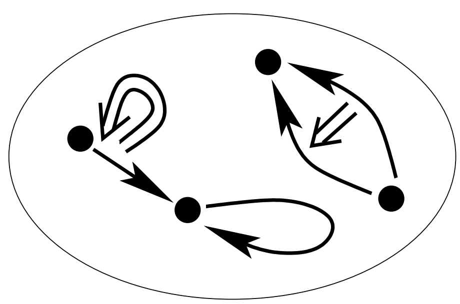
Using n-categories, instead of starting with a set of things:



we can now start with a category of things and *processes*:

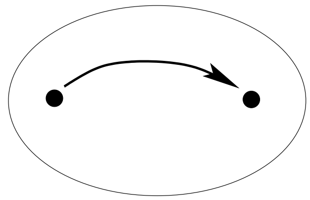


or a *2-category* of things, processes, and *processes between processes*:

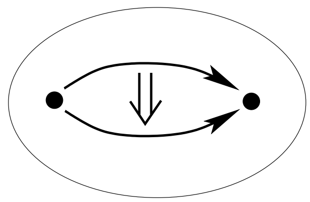


... and so on.

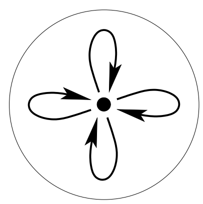
I'll illustrate this with examples from *higher gauge theory*. This describes not only how particles transform as they move along paths in spacetime:



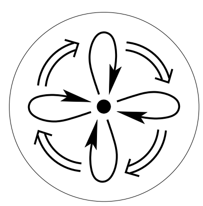
but also how strings transform as they trace out surfaces:



... and so on. Where ordinary gauge theory uses *groups*, which are special categories:



higher gauge theory uses *2-groups*:



which are special 2-categories. Where ordinary gauge theory uses *bundles*, higher gauge theory uses *2-bundles*. Everything gets 'categorified'!

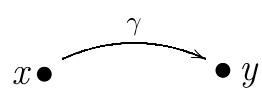
But first let's back up a bit....

The Fundamental Groupoid

Defining the fundamental group of a space X requires us to pick a basepoint $* \in X$. This is a bit *ad hoc*, and no good when X has several components.

Sometimes it's better to use the **fundamental groupoid** of X. This is the category $\Pi_1(X)$ where:

- objects are points of X: $\bullet x$
- morphisms are homotopy classes of paths in X:



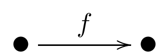
We compose homotopy classes of paths in the obvious way. Composition is associative, and every point has an identity path $1_x \colon x \to x$.

In short: *take the pictures seriously!*

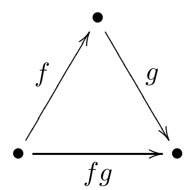
Eilenberg–Mac Lane Spaces

Conversely, the **nerve** of a groupoid G is a simplicial set with one vertex for each object:

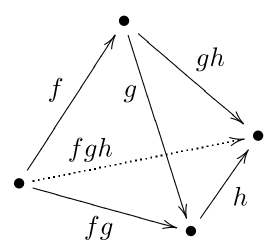
one edge for each morphism:



a triangle for each composable pair of morphisms:



a tetrahedron for each composable triple:



and so on! The **geometric realization** of this nerve is a space whose fundamental groupoid is equivalent to G. It's also a **homotopy 1-type**: all its homotopy groups above the 1st vanish. These facts characterize it — it's called the **Eilenberg–Mac Lane space** $K(G, 1)$.

Using this idea, one can show:

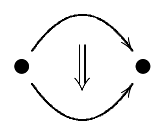
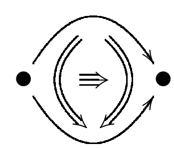
Homotopy 1-types are 'the same' as groupoids!

For starters: the fundamental groupoid is a complete invariant for homotopy 1-types.

Grothendieck's Dream

In a 600-page letter to Quillen, Grothendieck dreamt of a grand generalization. Categories should be a special case of *n-categories*. Say an n-category is an 'n-groupoid' if every j-morphism ($j < n$) is invertible up to a $(j+1)$-morphism, and n-morphisms are invertible on the nose.

Every space X should have a 'fundamental n-groupoid', $\Pi_n(X)$, where:

- objects are points of X: $\bullet$
- morphisms are paths in X: $\bullet \longrightarrow \bullet$
- 2-morphisms are paths of paths in X: 
- 3-morphisms are paths of paths of paths in X: 
- etcetera...

and we take homotopy classes only at the nth level.

A space is a **homotopy *n*-type** if its homotopy groups above the nth all vanish. Grothendieck dreamt that:

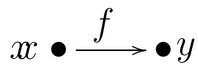
Homotopy n-types are 'the same' as n-groupoids!

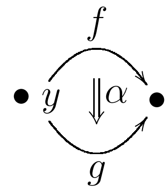
In 2005, Denis-Charles Cisinski made this precise and proved it using Batanin's definition of n-category.

2-Categories

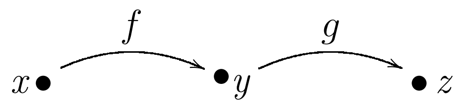
Try $n = 2$. A **weak 2-category** or **bicategory** has:

objects **morphisms** **2-morphisms**

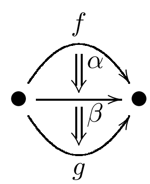


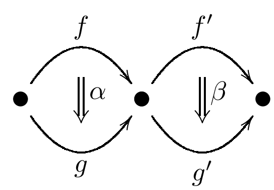


We can compose morphisms as usual:

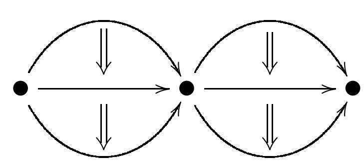


We can compose 2-morphisms both **vertically** and **horizontally**:





Vertical composition is associative and has left/right units. The **interchange law** holds, meaning the two ways of reading this agree:



Composition of morphisms satisfies the usual laws *up to natural 2-isomorphisms*: the **associator**:

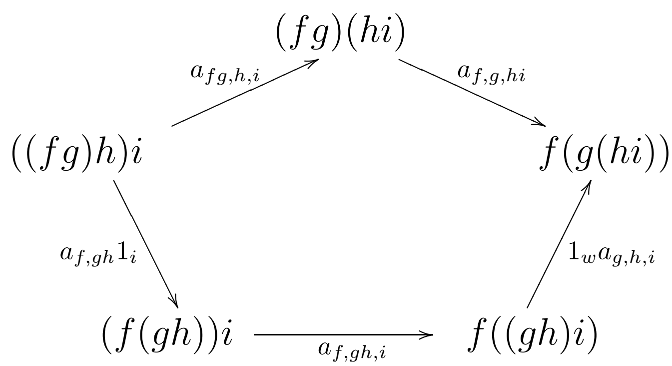
$$a_{f,g,h} \colon (fg)h \Rightarrow f(gh)$$

and **left and right unitors**:

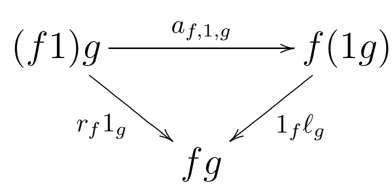
$$\ell_f \colon 1f \Rightarrow f$$

$$r_x \colon f1 \Rightarrow f$$

Finally, these must obey the **pentagon identity**:



and **triangle identity**:

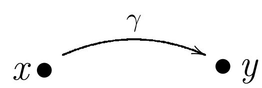


A 2-category is **strict** if the associator and left/right unitors are all identity 2-morphisms.

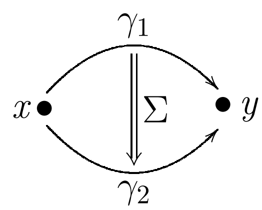
Classifying Homotopy 2-Types

A **2-groupoid** is a 2-category where 2-morphisms are invertible and morphisms are invertible *up to 2-morphisms.* Every space X has a **fundamental 2-groupoid** $\Pi_2(X)$, where:

- objects are points of X: $\bullet x$
- morphisms are paths in X:



- 2-morphisms are homotopy classes of paths-of-paths in X:

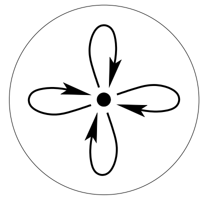


This is a complete invariant for homotopy 2-types, and we can go back by taking the geometric realization of the nerve.

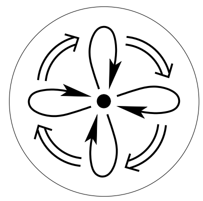
So: *to classify homotopy 2-types, classify 2-groupoids!*

Classifying 2-Groupoids

Just as a group is a groupoid with one object:



a **2-group** is a 2-groupoid with one object:



We may also think of it as a category with multiplication and inverses.

Every 2-groupoid is equivalent to a disjoint union of 2-groups! Moreoever,

Theorem (Joyal–Street). 2-groups are classified up to equivalence by isomorphism classes of (G, A, ρ, a) consisting of:

- a group G,
- an abelian group A,
- an action ρ of G as automorphisms of A,
- an element $[a]$ in the group cohomology $H^3(G, A)$.

If our 2-group comes from a pointed space X, then $G = \pi_1(X)$ and $A = \pi_2(X)$.

Lie 2-Algebras

Topology is more fun on manifolds: we can differentiate, and do *gauge theory*. Combining this with n-categories we get *higher gauge theory*.

For starters, we can define 'Lie 2-groups', and these have 'Lie 2-algebras'. Very roughly, a Lie 2-algebra is a category L with a vector space of objects, a vector space of morphisms and a bracket *functor*:

$$[\cdot, \cdot] \colon L \times L \to L$$

that satisfies the Jacobi identity up to a natural isomorphism, the **Jacobiator**:

$$J_{x,y,z} \colon [[x, y], z] \to [x, [y, z]] + [[x, z], y],$$

which must satisfy a certain identity of its own.

Theorem. Lie 2-algebras are classified up to equivalence by isomorphism classes of $(\mathfrak{g}, \mathfrak{a}, \rho, J)$ consisting of:

- a Lie algebra $\mathfrak{g}$,
- an abelian Lie algebra $\mathfrak{a}$,
- an action ρ of $\mathfrak{g}$ as derivations of $\mathfrak{a}$,
- an element $[J]$ in the Lie algebra cohomology $H^3(\mathfrak{g}, \mathfrak{a})$.

Just like the classification of 2-groups!

My Favorite Lie 2-Groups

Let's use these classifications to get nice examples.

If $\mathfrak{g}$ is a real simple Lie algebra, and $\mathfrak{a} = \mathbb{R}$ equipped with the trivial action of $\mathfrak{g}$, then

$$H^3(\mathfrak{g}, \mathfrak{a}) = \mathbb{R}$$

with this nontrivial 3-cocycle:

$$\begin{array}{rcl} \nu\colon \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} & \to & \mathbb{R} \\ x \otimes y \otimes z & \mapsto & \langle [x, y], z \rangle \end{array}$$

So, *every simple Lie algebra* $\mathfrak{g}$ *has a 1-parameter deformation* $\mathfrak{g}_k$ *in the world of Lie 2-algebras!* Here $k \in \mathbb{R}$ measures the nontriviality of the Jacobiator.

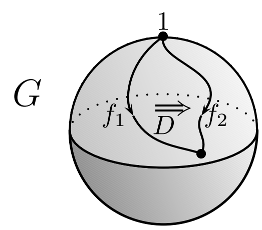
Do these Lie 2-algebras have corresponding Lie 2-groups?

Not in any *easy* sense — but morally speaking, *yes!*

And, they're related to the math of string theory.

Theorem. For any $k \in \mathbb{Z}$, there is an infinite-dimensional smooth 2-group $\mathcal{P}_k G$ whose Lie 2-algebra $\mathcal{P}_k \mathfrak{g}$ is equivalent to $\mathfrak{g}_k$.

An object of $\mathcal{P}_k G$ is a smooth path $f\colon [0, 2\pi] \to G$ starting at the identity. A morphism from f_1 to f_2 is an equivalence class of pairs (D, α) consisting of a disk D going from f_1 to f_2 together with $\alpha \in \mathrm{U}(1)$:



Any two such pairs (D_1, α_1) and (D_2, α_2) have a 3-ball B whose boundary is $D_1 \cup D_2$. The pairs are equivalent when

$$\exp\left(2\pi i k \int_B \nu\right) = \alpha_2/\alpha_1$$

where ν is the left-invariant closed 3-form on G with

$$\nu(x, y, z) = \langle [x, y], z \rangle$$

and $\langle \cdot, \cdot \rangle$ is the smallest invariant inner product on $\mathfrak{g}$ such that ν gives an integral cohomology class.

Theorem. The morphisms in $\mathcal{P}_k G$ starting at the constant path form the level-k central extension of the loop group ΩG:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \longrightarrow \Omega G \longrightarrow 1$$

$\mathcal{P}_k G$ and the String Group

The **nerve** of a smooth 2-group G is a simplicial smooth group. When we take its **geometric realization** we get a smooth group $|G|$.

Theorem. For any $k \in \mathbb{Z}$, there is a short exact sequence of smooth groups:

$$1 \longrightarrow \mathcal{L}_k G \longrightarrow \mathcal{P}_k G \longrightarrow G \longrightarrow 1$$

This gives a short exact sequence of smooth groups:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & |\mathcal{L}_k G| & \longrightarrow & |\mathcal{P}_k G| & \longrightarrow & G & \longrightarrow & 1 \\ & & \simeq \downarrow & & & & & & \\ & & K(\mathbb{Z},2) & & & & & & \end{array}$$

We have

$$\pi_3(|\mathcal{P}_k G|) \cong \mathbb{Z}/k\mathbb{Z}$$

and when $k = \pm 1$,

$$|\mathcal{P}_k G| \simeq \widehat{G},$$

which is the topological group obtained by killing the third homotopy group of G.

When $G = \mathrm{Spin}(n)$, $\widehat{G}$ is called $\mathrm{String}(n)$:

$$\mathrm{String}(n) \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to \mathrm{O}(n)$$

Next time we'll start doing gauge theory with 2-groups as 'gauge groups'.