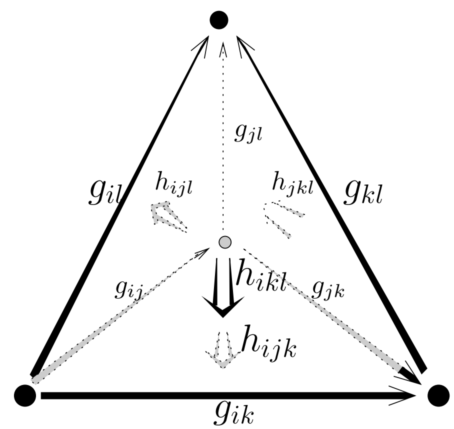
Classifying Spaces for Topological 2-Groups

John C. Baez and Danny Stevenson

University of Illinois at Urbana–Champaign,
Oct. 23, 2008



for online references, see:

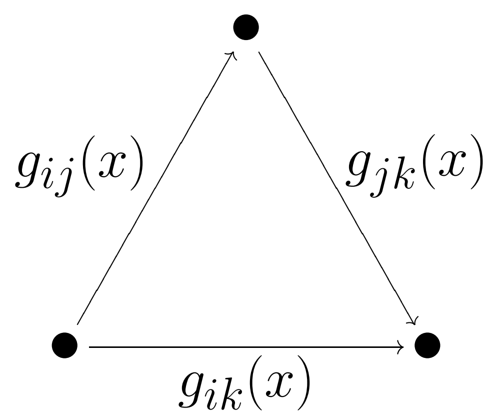
http://math.ucr.edu/home/baez/barcelona/

Čech Cohomology for Bundles

If G is a topological group and M is a topological space, we can describe a principal G-bundle $P \to M$ using a **Čech cocycle**. This consists of an open cover $\mathcal{U} = \{U_i\}$ of M together with **transition functions**

$$g_{ij} \colon U_i \cap U_j \to G$$

such that

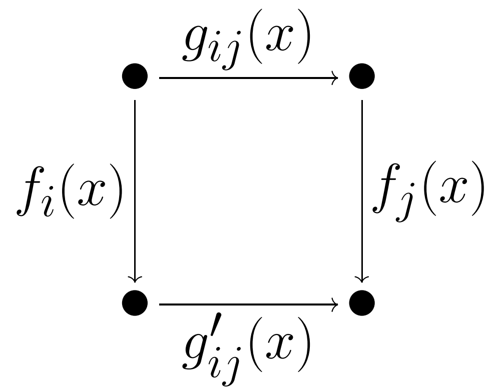


commutes for all $x \in U_i \cap U_j \cap U_k$.

Two Čech cocycles define isomorphic bundles iff they are **cohomologous**, meaning there are functions

$$f_i \colon U_i \to G$$

such that



commutes for all $x \in U_i \cap U_j$.

The set of cohomology classes of Čech cocycles is called $\check{H}(\mathcal{U}, G)$. Taking the inverse limit as we refine the open cover, we obtain the (first) **Čech cohomology** of M with coefficients in G:

$$\check{H}(M, \mathcal{G}) = \varprojlim_{\mathcal{U}} \check{H}(\mathcal{U}, G)$$

There is a bijection between $\check{H}(M, G)$ and the set of isomorphism classes of principal G-bundles over M.

A Famous Old Theorem

Here is the result we'd like to categorify — a result first due to Milnor but polished by Steenrod, Segal, Milgram and May:

Thm. Let G be a well-pointed topological group. Let BG, the **classifying space** of G, be the geometric realization of the nerve of G. Then for any paracompact Hausdorff space M, there is a bijection

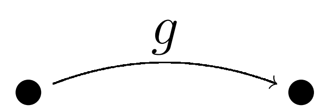
$$[M, BG] \cong \check{H}(M, G)$$

(A topological group G is **well-pointed** if $1 \in G$ has a neighborhood of which it is a deformation retract.)

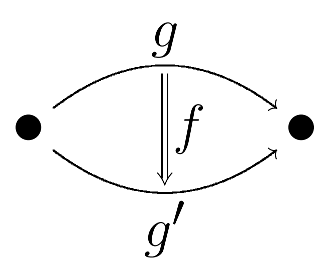
Topological 2-Groups

Defn. A **2-group** is a category with a group of objects and a group of morphisms, such that all the category operations are group homomorphisms.

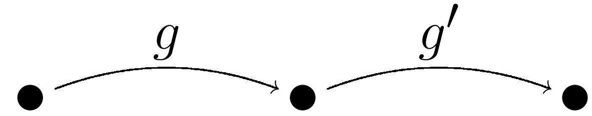
We draw the objects like this:



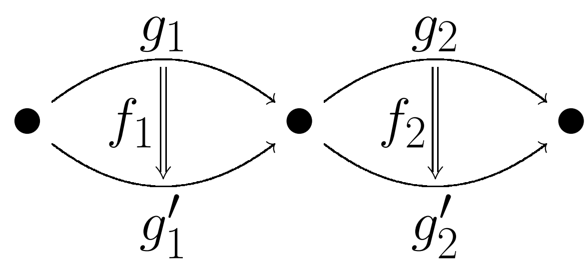
and the morphisms like this:



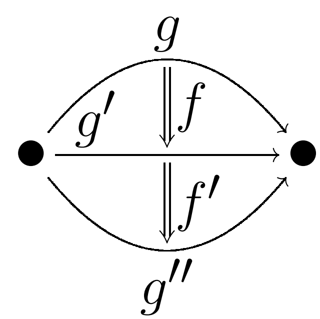
We can multiply objects:



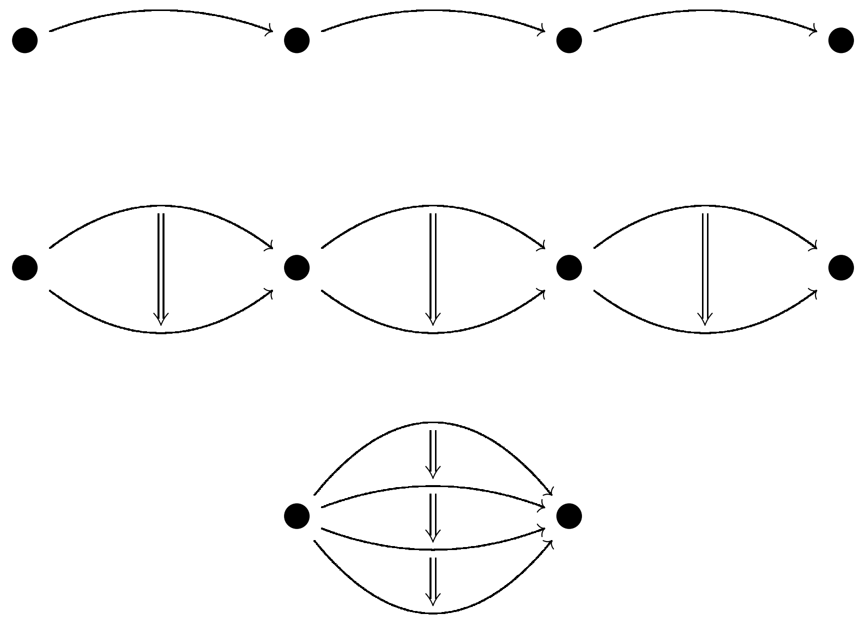
multiply morphisms:



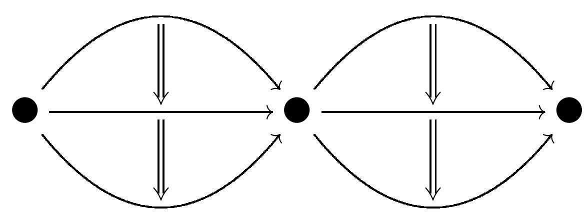
and compose morphisms:



All three operations have a unit and inverses. All three are associative, so these are well-defined:



Finally, the **interchange law** holds:



is well-defined.

Defn. A **topological 2-group** is a 2-group with a topological group of objects and a topological group of morphisms, for which all the 2-group operations are continuous.

Two examples important in string theory:

- Any abelian topological group A gives a topological 2-group $A[1]$ with one object and A as morphisms.
- Any simply-connected compact simple Lie group G gives a topological 2-group $\mathrm{String}(G)$.

Čech Cohomology for 2-Bundles

The Basic Idea: a Čech cocycle with coefficients in a topological 2-group $\mathcal{G}$ is a recipe for building a 'principal $\mathcal{G}$-2-bundle' over M using 'transition functions'. Two such 2-bundles will be 'equivalent' when their cocycles are cohomologous.

We won't define '2-bundles' here: see Toby Bartels' thesis or the work of Baas, Bökstedt and Kro.

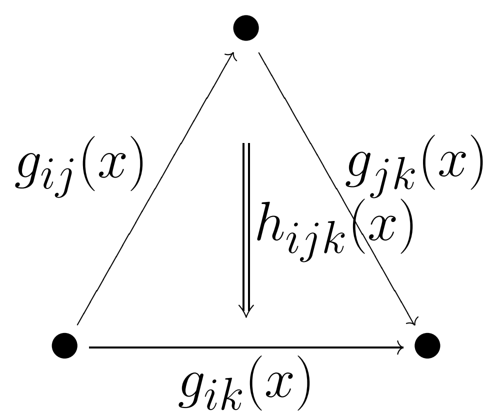
Instead, let's go straight to Čech cohomology!

Let $\mathcal{U} = \{U_i\}$ be an open cover of a topological space M, and let $\mathcal{G}$ be a topological 2-group

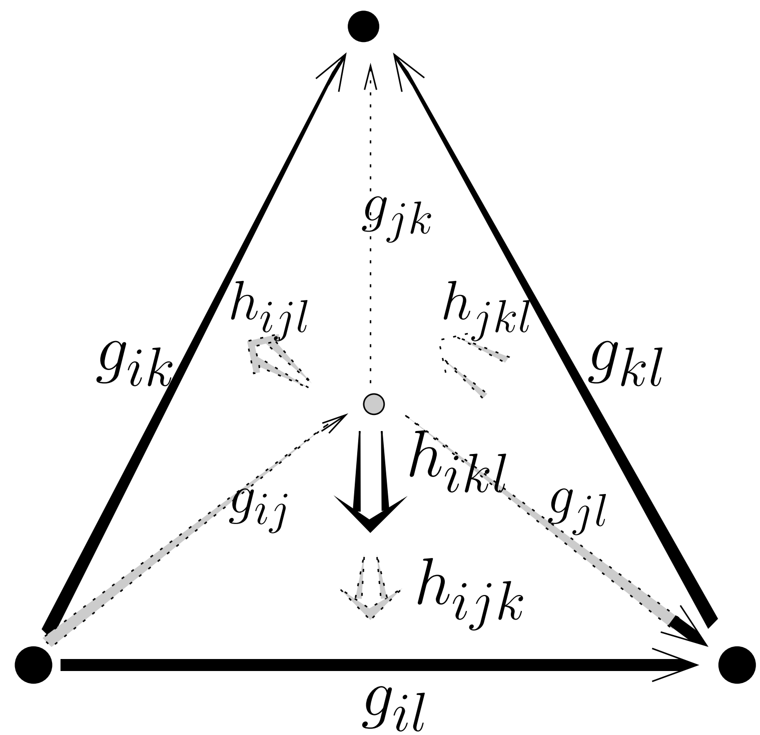
Defn. A **Čech cocycle** with coefficients in $\mathcal{G}$ consists of the following data:

For each $x \in U_i \cap U_j$, an object $g_{ij}(x)$ in $\mathcal{G}$ depending continuously on x.

For each $x \in U_i \cap U_j \cap U_k$, a morphism $h_{ijk}(x)$ in $\mathcal{G}$ depending continuously on x that fills in this triangle:



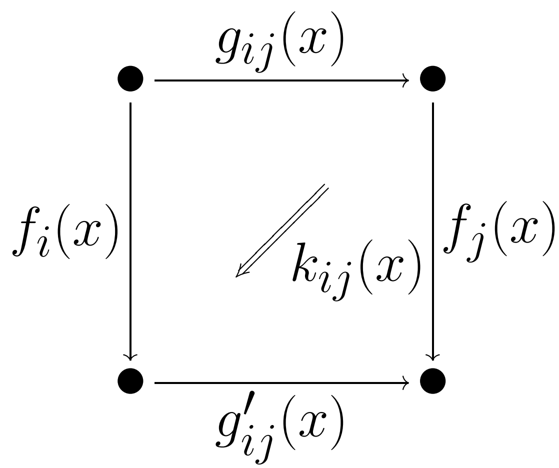
Finally, the h_{ijk} must make these tetrahedra commute:



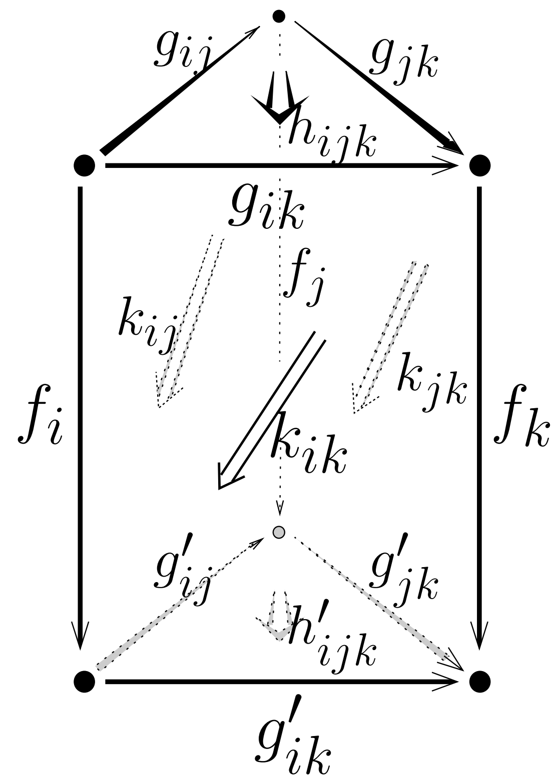
Defn. Two Čech cocycles (g, h) and (g', h') are **cohomologous** if we have the following data:

For each $x \in U_i$, an object $f_i(x)$ of $\mathcal{G}$ depending continuously on x.

For each $x \in U_i \cap U_j$, a morphism $k_{ij}(x)$ in $\mathcal{G}$ depending continuously on x that fills in this square:



Finally, the k_{ij} must make these prisms commute:



Defn. Let $\check{H}(\mathcal{U}, \mathcal{G})$ be the set of cohomology classes of Čech cocycles. We define the **Čech cohomology** of M with coefficients in $\mathcal{G}$ to be the inverse limit as we refine the cover:

$$\check{H}(M, \mathcal{G}) = \varprojlim_{\mathcal{U}} \check{H}(\mathcal{U}, \mathcal{G})$$

Categorifying That Famous Old Theorem

Thm. Suppose $\mathcal{G}$ is a well-pointed topological 2-group and M is a paracompact Hausdorff space admitting good covers. Then there is a bijection

$$\check{H}(M, \mathcal{G}) \cong [M, B|\mathcal{G}|]$$

where the topological group $|\mathcal{G}|$ is the geometric realization of the nerve of $\mathcal{G}$. So, we call $B|\mathcal{G}|$ the **classifying space** of $\mathcal{G}$.

(A topological 2-group G is **well-pointed** if both the topological groups in its corresponding crossed module are well-pointed. An open cover is **good** if each nonempty finite intersection of open sets in the cover is contractible.)

A Corollary:
Bundles vs. 2-Bundles

Cor. There is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over X
- elements of the Čech cohomology $\check{H}(M, \mathcal{G})$
- homotopy classes of maps $f\colon X \to B|\mathcal{G}|$
- elements of the Čech cohomology $\check{H}(M, |\mathcal{G}|)$
- isomorphism classes of principal $|\mathcal{G}|$-bundles over X.

Characteristic Classes for String(G)-2-bundles

Now suppose G is a compact simply-connected simple Lie group and $\mathrm{String}(G)$ is its string 2-group:

Thm. There is a short exact sequence of topological groups

$$1 \longrightarrow K(\mathbb{Z},2) \longrightarrow B|\mathrm{String}(G)| \xrightarrow{\ p\ } G \longrightarrow 1$$

where p is a fibration. This exhibits $B|\mathrm{String}(G)|$ as the 3-connected cover of G.

Matt Ando helped us show the following:

Cor. The homomorphism

$$B|\mathrm{String}(G)| \xrightarrow{p} G$$

gives an algebra homomorphism:

$$H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|\mathrm{String}(G)|, \mathbb{R})$$

This is onto, with kernel generated by the 'second Chern class' $c_2 \in H^4(BG, \mathbb{R})$.

So: the real characteristic classes of String(G)-2-bundles are just like those of G-bundles, but with the second Chern class killed!