Controllability and observability: diagrams and duality

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Outline

- Examples and motivation
- Creating new diagrams from old
- Controllability and observability

Motivation (Why diagrams?)

In physics, the informal use of graphical calculi for symmetric monoidal categories goes back to Penrose, as early as 1971. But it wasn't until 1991 that Joyal and Street proved the soundness of working with diagrams in a symmetric monoidal category. Selinger went further in 2007. Selinger extended their results to graphical calculi for dagger compact closed categories.

This is important because the categories that appear in control systems that I will talk about are dagger compact closed categories.

Motivation (Conceptual simplification)

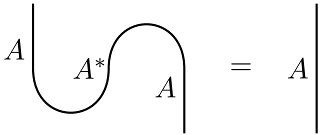
The following equation is one of those used in the definition of a compact closed category:

$$\lambda_A^{-1} \circ (\epsilon_A \otimes A) \circ \alpha^{-1}_{A,A^*,A} \circ (A \otimes \eta_A) \circ \rho_A = \mathrm{id}_A$$

Using semicolon notation for composition:

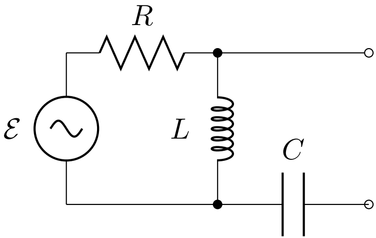
$$\rho_A ; (A \otimes \eta_A) ; \alpha^{-1}_{A,A^*,A} ; (\epsilon_A \otimes A) ; \lambda_A^{-1} = \mathrm{id}_A$$

The same equation, written in 2d:



Motivation (Communication)

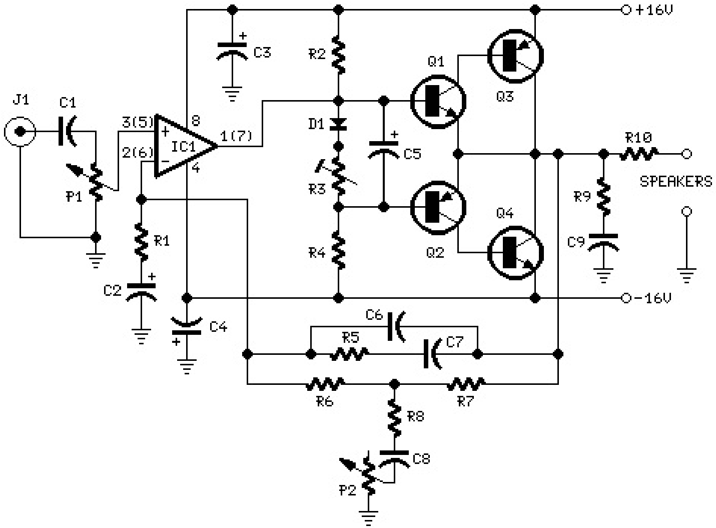
How would you describe this electric circuit if you had no 2d language?



Examples

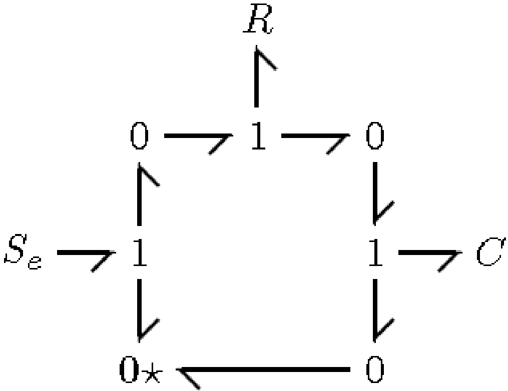
- Circuit diagrams
- Bond graphs
- Block diagrams
- Petri nets
- ZX calculus / ZW calculus
- Feynman diagrams
- Graphical linear algebra
- Natural language

Circuit diagrams



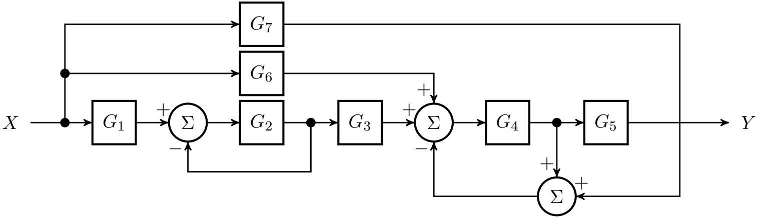
From math.ucr.edu/home/baez/control/control_talk_erlangen.pdf

Bond graphs

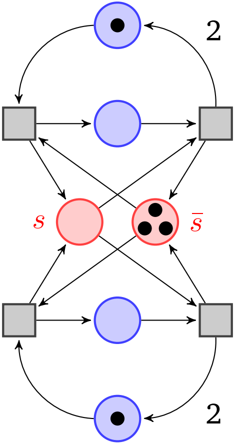


From en.wikipedia.org/wiki/Bond_graph

Block diagrams

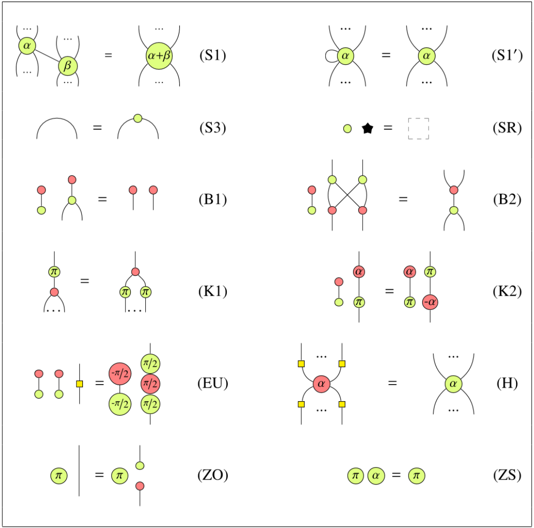


Petri nets



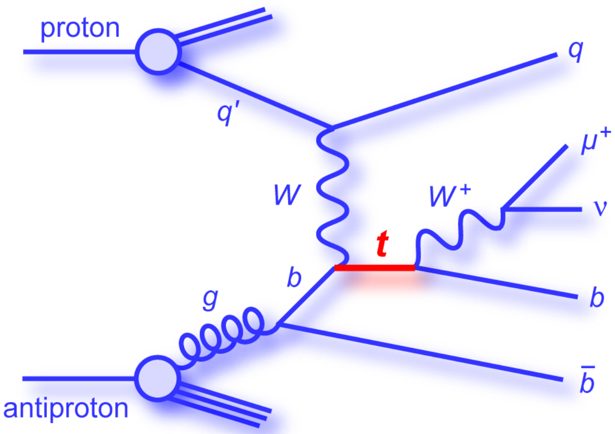
From the PGF manual, tutorial chapter

ZX calculus



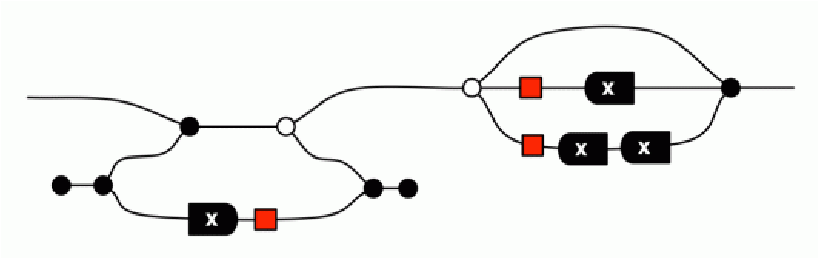
From *A Simplified Stabilizer ZX-calculus* (arXiv:1602.04744)

Feynman diagrams



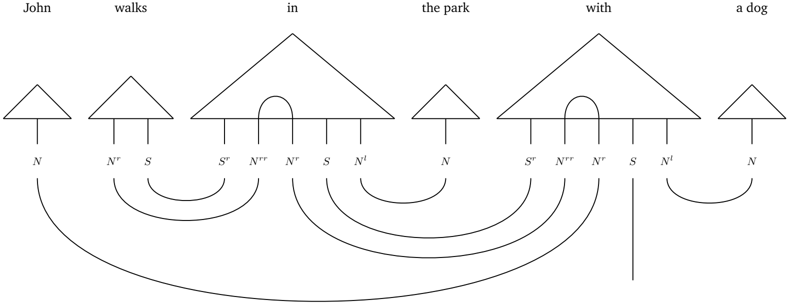
From math.ucr.edu/home/baez/control/control_talk_erlangen.pdf

Graphical linear algebra



From GraphicalLinearAlgebra.net

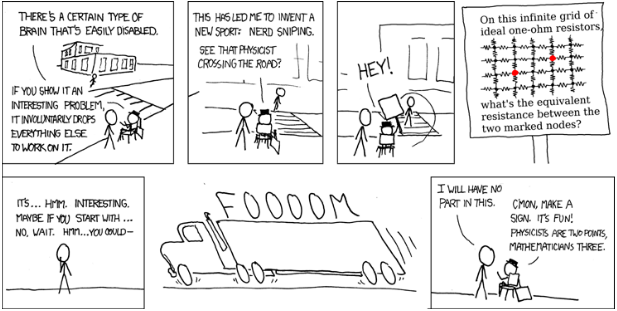
Natural language



From *Compositional Distributional Semantics with Compact Closed Categories and Frobenius Algebras* (arXiv:1505.00138)

Useful

2d reasoning is not just for nerd sniping *à la* xkcd/356:

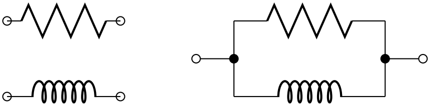


It also aids in

- communication
- intuition
- calculation
- understanding dualities

Building diagrams

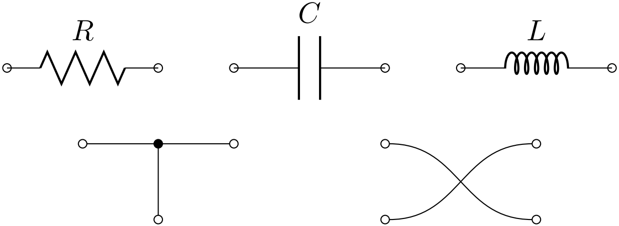
In 1d everything has to be done in series. By adding a second dimension, things can be done in parallel. Series compositions are usually denoted $\circ$, while parallel compositions are usually denoted $\otimes$. Note, though, that parallel composition of circuits is not the same as parallel circuits:



If we know all of the fundamental building block diagrams, any diagram can be built from series and parallel compositions of the building blocks.

Building diagrams

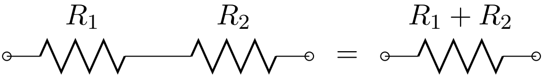
For linear passive electric circuits, those building blocks are resistors, capacitors, inductors, junctions, and a symmetric braiding:



where R, L, and C are positive real numbers.
The symmetric braiding can be taken as implicit when the category of diagrams is a PROP.

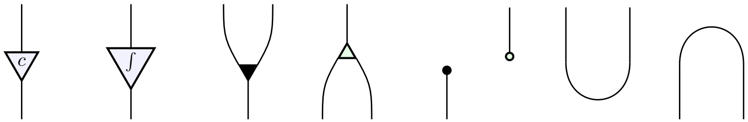
Simplifying diagrams

Often we want to be able to say two diagrams are “the same”, possibly preferring one over the other if it is particularly nice for calculations. Using **rewrite rules**, complicated-appearing diagrams can be reduced to simpler-appearing diagrams, and *vice versa*.



Having a complete set of rewrite rules means any two equivalent diagrams can be connected by a chain of local applications of those rules. The rewrite rules are **confluent** if they are one-way, and local applications of those rules to any two equivalent diagrams will always lead to the same final diagram.

Control systems building blocks

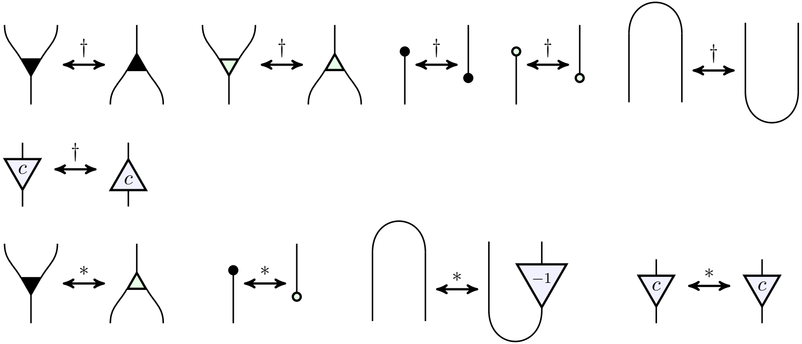


where $c \in k$.

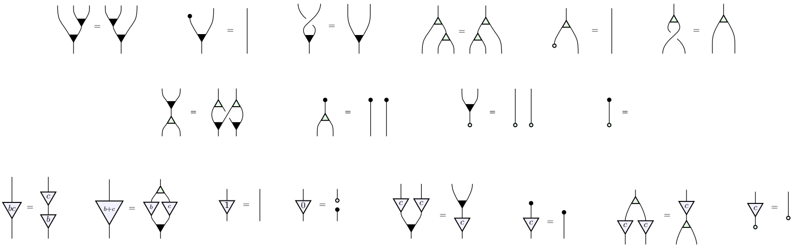
- Multiplication by a scalar
- Integration
- Addition
- Duplication
- Zero
- Delete
- Cup
- Cap

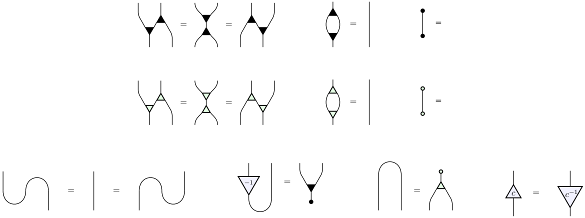
Dualities

Two contravariant endofunctors give two different dualities on this equational system, denoted here with $\dagger$ and $*$. Roughly, $\dagger$ can be thought of as 'inverse' and $*$ can be thought of as 'transpose'.



Rewrite rules



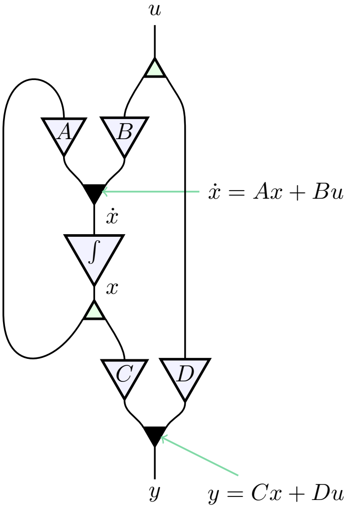


Controllability and Observability

For a system of matrix equations,

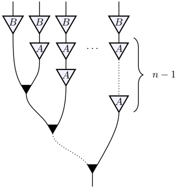
- Differential:
  $\dot{x} = Ax + Bu$
- Linear:
  $y = Cx + Du$

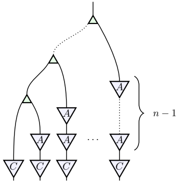
we can find a signal flow diagram that 'contains' these equations.



u is the input vector ($\in \mathbb{R}^m$)
x is the 'state' vector ($\in \mathbb{R}^n$)
y is the output vector ($\in \mathbb{R}^p$).

Controllability and Observability

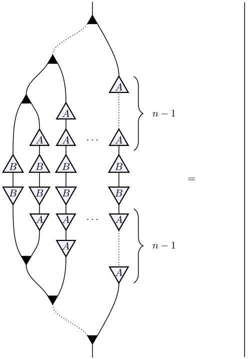




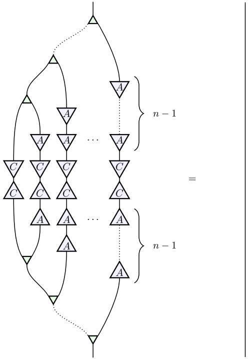
is an epimorphism
for a controllable system

is a monomorphism
for an observable system

Controllability and Observability

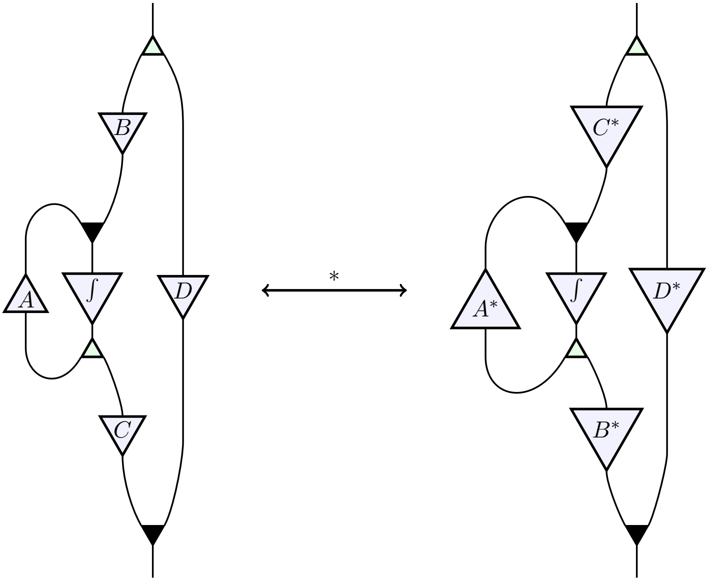


for a controllable system



for an observable system

Controllability and Observability



Future work

Much is still missing in this story:

- Stability
- Nonlinearities
- Time dependence
- Continuous / discrete hybrid systems
- Relational signal flow diagrams (A, B, C, and/or D as (linear) relations)

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Paweł Sobociński's Graphical Linear Algebra blog: http://graphicallinearalgebra.net