Coarse-graining open Markov processes

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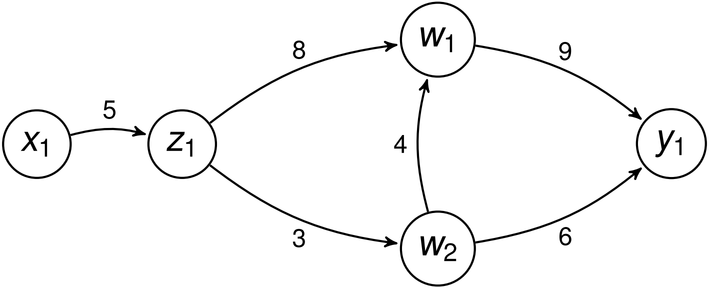
A Markov process looks something like this:



Definition

A **Markov process** on a finite set S of states consists of a map $H\colon S \times S \to \mathbb{R}$ such that

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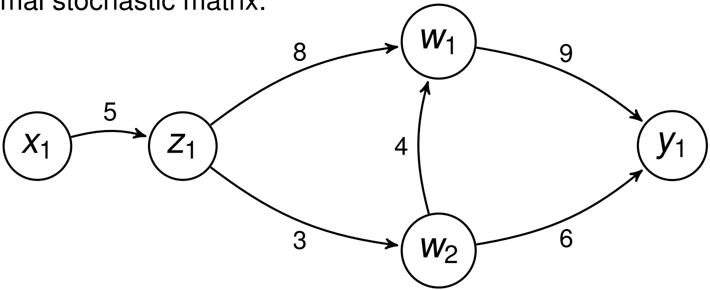
Definition

An **infinitesimal stochastic matrix** is a square matrix H where each entry $H_{i,j}$ is non-negative for $i \neq j$ and the sum of the entries in each column is 0.

We can thus think of a Markov process on a finite set S as a $|S| \times |S|$ infinitesimal stochastic matrix.



$$H = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 \\ 5 & -11 & 0 & 0 & 0 \\ 0 & 8 & -9 & 4 & 0 \\ 0 & 3 & 0 & -10 & 0 \\ 0 & 0 & 9 & 6 & 0 \end{bmatrix}$$

Definition

An **open Markov process** is a cospan of finite sets where the apex is equipped with a Markov process.

$$X \xrightarrow{i} S \xleftarrow{o} Y \qquad H\colon S \times S \to \mathbb{R}$$

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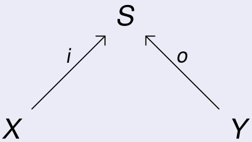

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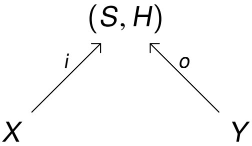
For brevity, we will denote an open Markov process as:



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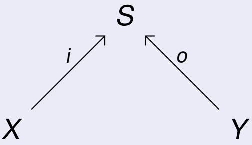

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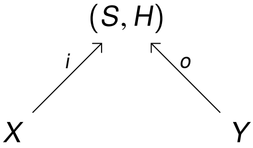
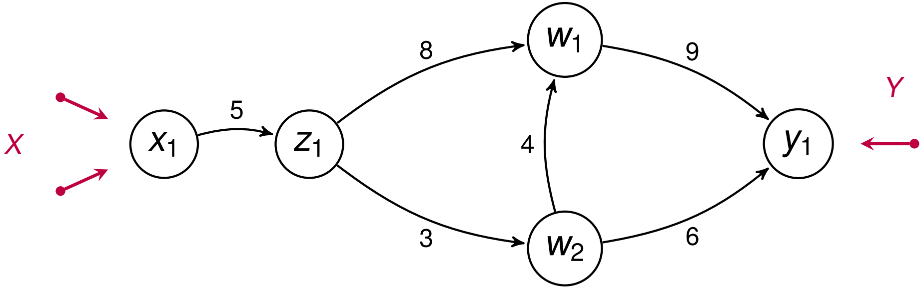
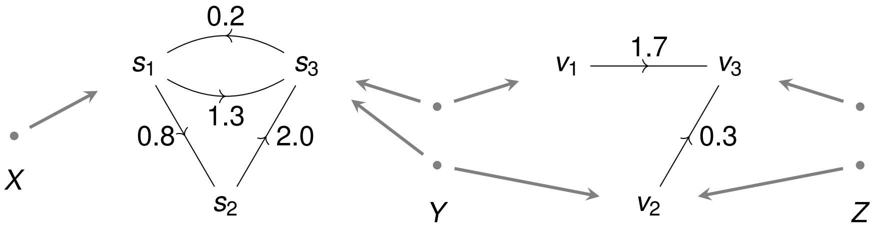


Open Markov processes will constitute the morphisms in a 'double category'.

Here's an example of an open Markov process:



We can compose an open Markov process whose outputs coincide with the inputs of another:



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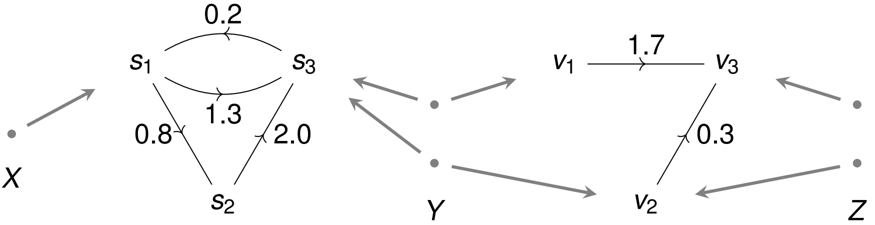
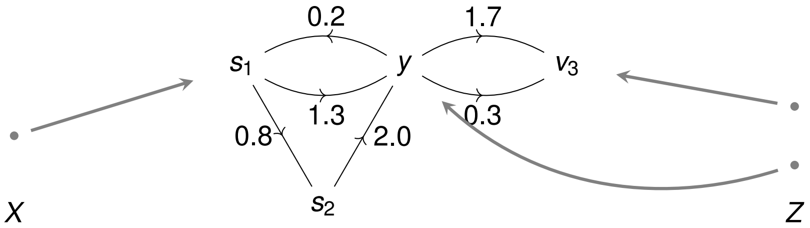


$$H_S = \begin{bmatrix} -2.1 & 0 & 0.2 \\ 0.8 & -2 & 0 \\ 1.3 & 2 & -0.2 \end{bmatrix} \qquad H_V = \begin{bmatrix} -1.7 & 0 & 0 \\ 0 & -0.3 & 0 \\ 1.7 & 0.3 & 0 \end{bmatrix}$$

To obtain another:



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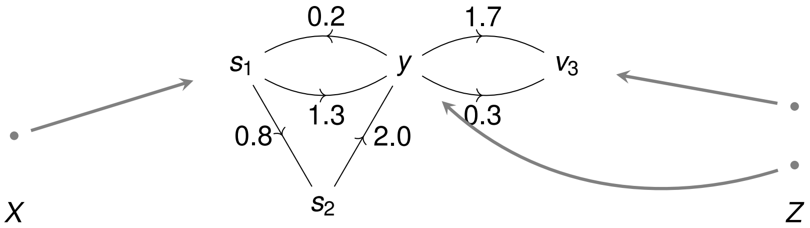

$$H_S \odot H_V = \begin{bmatrix} -2.1 & 0 & 0.2 & 0 \\ 0.8 & -2 & 0 & 0 \\ 1.3 & 2 & -2.2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

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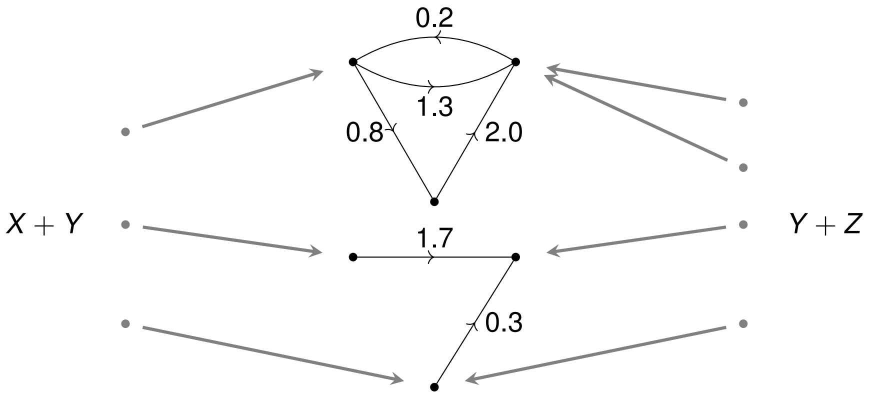
we can obtain another infinitesimal stochastic map

$$H_S \odot H_V \colon (S +_Y V) \times (S +_Y V) \to \mathbb{R}.$$

We can also tensor two open Markov processes by placing them side by side:



The two infinitesimal stochastic matrices for these two open Markov processes are given respectively by

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The infinitesimal stochastic matrix for the tensor of these two open Markov processes is then given by the direct sum of the above two infinitesimal stochastic matrices.

$$H_S \oplus H_V = \begin{bmatrix} -2.1 & 0 & 0.2 & 0 & 0 & 0 \\ 0.8 & -2 & 0 & 0 & 0 & 0 \\ 1.3 & 2 & -0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.3 & 0 \\ 0 & 0 & 0 & 1.7 & 0.3 & 0 \end{bmatrix}$$

We ultimately build a ‘double-category’, which has figures like this:

$$\begin{array}{ccc} A & \xrightarrow{\;M\;} & B \\ {\scriptstyle f}\downarrow & \Downarrow a & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\;N\;]{} & D \end{array}$$

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and morphisms between horizontal 1-cells, called 2-morphisms, here denoted as a.

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$\Downarrow$ Coarse-grainings

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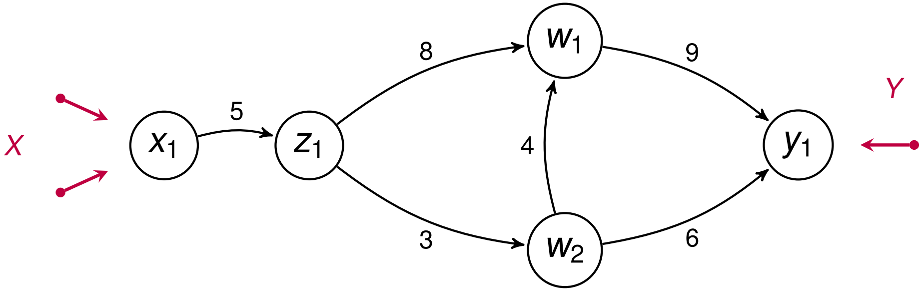
A coarse-graining should look something like this:



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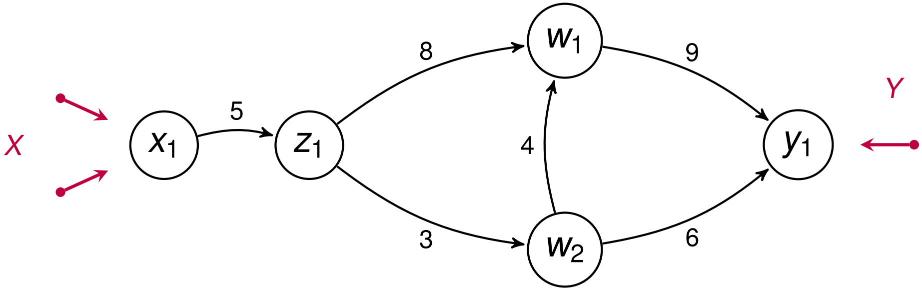

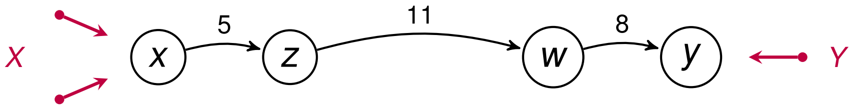

$$H' = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 5 & -11 & 0 & 0 \\ 0 & 11 & -8 & 0 \\ 0 & 0 & 8 & 0 \end{bmatrix}$$

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Definition

A **stochastic map** $s\colon S' \rightsquigarrow S$ is a map $s\colon S \times S' \to \mathbb{R}$ such that

$\sum_{x \in S} s(x, y) = 1$ for each $y \in S'$ and

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Since a function is a special case of a stochastic map, and a stochastic map can be thought of as a stochastic matrix, we can have inclusion of categories:

$$\mathrm{FinSet} \subseteq \mathrm{FinStoch} \subseteq \mathrm{Mat}(\mathbb{R}).$$

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Theorem

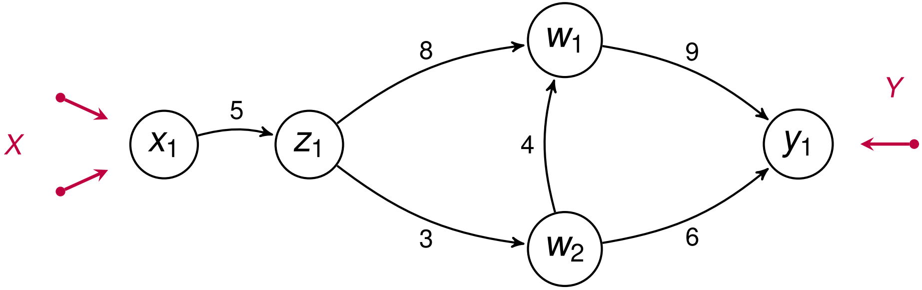
Let $p\colon S \to S'$ be a surjection with stochastic section $s\colon S' \rightsquigarrow S$ and $H\colon S \times S \to \mathbb{R}$ an infinitesimal stochastic matrix. Then $H' = pHs\colon S' \times S' \to \mathbb{R}$ is an infinitesimal stochastic matrix.

Definition

Given an open Markov process $X \xrightarrow{i} (S,H) \xleftarrow{o} Y$, a **coarse graining** is given by (f,p,g,s) where $f\colon X \to X'$ and $g\colon Y \to Y'$ are bijections, $p\colon S \to S'$ is a surjection and $s\colon S' \rightsquigarrow S$ is a stochastic section of p such that the following (underlying) diagram commutes in FinSet

$$
\begin{array}{ccccc}
X & \longrightarrow & (S,H) & \longleftarrow & Y \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle (p,s)} & & \big\downarrow{\scriptstyle g} \\
X' & \longrightarrow & (S',H') & \longleftarrow & Y'.
\end{array}
$$

with $H' = pHs$.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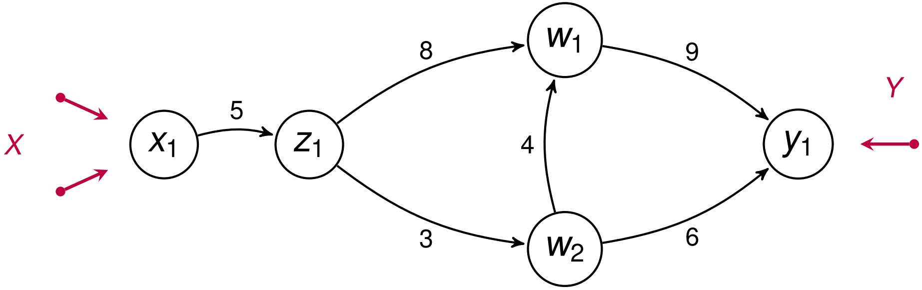

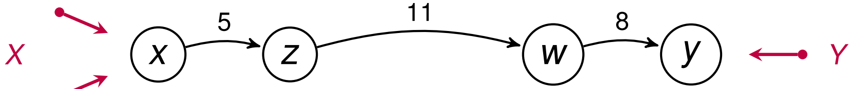

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One of the main results in the paper *Coarse-graining open Markov processes* is the following:

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For an open Markov process $X \to (S,H) \leftarrow Y$*:*

$$\blacksquare\big(X \to (S,H) \leftarrow Y\big) \subseteq \mathbb{R}^X \oplus \mathbb{R}^X \oplus \mathbb{R}^Y \oplus \mathbb{R}^Y$$

consisting of all 4-tuples (i^*v, I, o^*v, O) *where* $v \in \mathbb{R}^S$ *is some steady state with inflows* $I \in \mathbb{R}^X$ *and outflows* $O \in \mathbb{R}^Y$.

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For a coarse-graining given by (f,p,g,s),

$$\blacksquare(f,p,g,s)\colon \mathbb{R}^X \oplus \mathbb{R}^X \oplus \mathbb{R}^Y \oplus \mathbb{R}^Y \to \mathbb{R}^{X'} \oplus \mathbb{R}^{X'} \oplus \mathbb{R}^{Y'} \oplus \mathbb{R}^{Y'}$$

is the linear map defined by

$(i^*(v), I, o^*(v), O) \mapsto (i'^*(p_*(v)), I, o'^*(p_*(v)), O) \subset \mathbb{R}^{X'} \oplus \mathbb{R}^{X'} \oplus \mathbb{R}^{Y'} \oplus \mathbb{R}^{Y'}$.

For more details, see our paper on the arXiv:

J. Baez and K. Courser, Coarse-graining open Markov processes. Available as arXiv:1710.11343.