Commonly Asked Questions

1. Solution to the in-class problem: Find the arc length function for the curve given by

\[ y = \ln(1 - x^2) \]

starting at \( x = 0 \).

Since we are going to find an arc length function, we begin by changing the variable \( x \) to \( t \), so that we will end up with a function of \( x \) in the end. Thus, we get

\[ y = \ln(1 - t^2). \]

Taking the derivative, we get

\[ \frac{dy}{dt} = \frac{-2t}{1 - t^2}, \]

and squaring it we get

\[ \left( \frac{dy}{dt} \right)^2 = \frac{4t^2}{(1 - t^2)^2}. \]

Adding one, find a common denominator and simplify as follows:

\[ 1 + \left( \frac{dy}{dt} \right)^2 = 1 + \frac{4t^2}{(1 - t^2)^2} = \frac{(1 - t^2)^2 + 4t^2}{(1 - t^2)^2} = \frac{1 - 2t^2 + t^4 + 4t^2}{(1 - t^2)^2} = \frac{1 + 2t^2 + t^4}{(1 - t^2)^2} = \frac{(1 + t^2)^2}{(1 - t^2)^2}. \]

Now, to get the arc length function starting at \( x = 0 \), we take

\[ s(x) = \int_0^x \sqrt{\frac{(1 + t^2)^2}{(1 - t^2)^2}} \, dt = \int_0^x \frac{1 + t^2}{1 - t^2} \, dt. \]
To take this integral, we need to change the form of this fraction. First, we can split up the numerator to get

\[
\frac{1 + t^2}{1 - t^2} = \frac{t^2}{1 - t^2} + \frac{1}{1 - t^2}.
\]

The first part needs to be divided out, using long division, and the result is

\[
\frac{t^2}{1 - t^2} = \frac{1}{1 - t^2} - 1.
\]

Thus, our integral can now be written as

\[
s(x) = \int_0^x \left( \frac{1}{1 - t^2} + \frac{1}{1 - t^2} - 1 \right) dt = \int_0^x \left( \frac{2}{1 - t^2} - 1 \right) dt = 2 \int_0^x \frac{1}{1 - t^2} dt - \int_0^x dt.
\]

To compute the first integral, check the formulas in the back of the book to find

\[
\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{u + a}{u - a} \right| + c.
\]

Here, we have \(a = 1\) and \(u = t\). Thus, computing both pieces we get

\[
s(x) = \left( \ln \left| \frac{t + 1}{t - 1} \right| - t \right) \bigg|_0^x = \ln \left| \frac{x + 1}{x - 1} \right| - x - \left( \ln \left| \frac{1}{1 - 1} \right| - 0 \right) = \ln \left| \frac{x + 1}{x - 1} \right| - x.
\]